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The authors investigate the possibility of utilizing the depreciation mechanism to provide incentives for undertaking investment in the real sector of the Russian economy. The proposed model of investor's behavior under risk and uncertainty considers a wide range of tax instruments The authors derive the optimal timing rule for investment and depreciation policy which maximizes tax payments into the regional budget. A comparative analysis of the former and new profit taxation systems is carried out. The authors discover interdependence of tax holidays and accelerated depreciation, and study the replacement of property tax by real estate tax. They investigate the possibilities of compensating for risk by reducing profit tax rate and changing depreciation policy, and prove the existence of risk zones for which these fiscal mechanisms are non-effective.

Keywords: Russia, tax system, investment project, uncertainty and risk, depreciation, tax exemptions.

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CONTENTS

NON-TECHNICAL SUMMARY

Apart from being used for fiscal purposes, the tax system can be used to stimulate the investment in the real sector of a transition economy. In terms of this task, one of the basic elements of the tax system is depreciation. It is used to transfer the cost of assets to the costs of the product. Thus, depreciation has a direct effect on the tax base for the corporate profit tax. Additionally, depreciation policy determines the residual cost of assets, which is the tax base of property tax. Hence, an increase in depreciation charges reduces the tax base of the property and income tax and can potentially act as a stimulus for investments. Acute interest in the depreciation mechanism may be also attributed to the fact that depreciation charges are, at present, an important source of capital budgeting for enterprises in the Russian economy.

The present study focuses on the modeling of the mechanism of accelerated depreciation that is aimed at attracting investments to new venture projects in the Russian real sector. Such models should be careful to include a certain number of factors.

First, there is an uncertainty related to the fluctuations of market prices for inputs and outputs, as well as investment resources that are needed for establishment of a firm. Consideration of this factor requires to model the cash flows of the future firm as a random process.

Second, unlike portfolio investments, investments in the new project are sunk costs, i.e. that they cannot be used for other purposes once the firm is set up.

And the third factor, which should also be taken into account, is the fiscal environment that the company is supposed to operate in. In this paper, the fiscal environment consists of corporate income tax, VAT, unified social tax, enterprise property tax, personal income tax, mechanisms of accelerated depreciation and tax holidays. Inclusion of the taxes listed above in the model makes if fully comply with the tax environment defined by the new Tax Code of the Russian Federation.

The behavior of potential investor is assumed to be rational. Observing the market prices, investor can decide either to invest in the project or postpone the decision until the situation becomes more favorable. When making a decision, the investor also considers various potentially adverse events that may result in partial losses of property and profit. The investor uses available information to choose the best moment for investment in order to maximize its net present value (NPV).

This study elaborates and analyzes a model of investor's behavior which takes into account all the factors mentioned above. The model is set up in continuous time. It is also assumed that the processes describing the dynamics of investment and the flows of value added are geometric Brownian motions. These assumptions allow us to employ the well known analytical methods of the theory of diffusion processes. Those methods were used in order to obtain an optimal investment rule and to explicitly construct the dependence of such a rule on the tax system's parameters.

The analysis revealed an unexpected phenomenon related to the simultaneous use of tax holidays and accelerated depreciation. In particular, we indicate the conditions under which an increase in depreciation charges (in conjunction with tax holidays) leads to a delay of the investor's arrival. Similarly, an increase in tax holidays under certain conditions can delay the investor's arrival. In other words, simultaneous usage of the two exemptions may entail negative effects.

The issue of compensation for risk process by tax exemptions was studies as well. It is demonstrated that there are critical values of the parameter defining the risk process. If they are exceeded, compensation for the risks taken by the investor either by decreasing the rate of profit tax or increasing depreciation charges will be impossible. This phenomenon can partly explain why there was no splash of investment after the adoption of the new profit tax law.

In this paper, the former system of profit taxation (that is characterized by allowance for tax exemptions such as tax holidays and accelerated depreciation) is compared to the new tax system that provide a significantly lower rate of profit tax, but no exemptions. It is shown that the new system is more preferable than the old one since it encourages investment, especially for firms with a high share of active assets.

The investor model served as a basis in order to obtain an optimal depreciation policy that maximizes expected tax revenues in the regional budget. This policy turned out to be beneficial for the federal budget and for the investor in the case of projects with a large part of active assets, average labor intensity (in term of wage per unit of value added) and moderate volatility.

In conclusion, in view of the announced property tax reform, it is especially interesting to study the replacement of the property tax by a real estate tax, which exists in most of the market economies. In Russia this replacement has been legally introduced as an experiment in Tver and Novgorod since 1997. The authors prove that such a replacement would encourage the investors to prefer capital-intensive projects whose share of active capital assets exceeds a certain critical value depending on parameters of the model.

1. INTRODUCTION

A long-term social and economic development program in a typical Russian region is closely tied with the implementation of few critical investment projects. Attracting investors to those projects is a key objective of such a program.

One way to solve this problem is the use of fiscal mechanisms such as investment incentives.

The present paper is dedicated to the analysis of the potential that the depreciation mechanisms have in attracting investment to new projects. Depreciation, as it is well known, is the economic mechanism for the transfer of the cost of assets to the cost of the product (Article 253 of Tax Code of the Russian Federation, or briefly TC RF hereinafter). Thus depreciation has a direct impact on the corporate profit tax base. The depreciation of assets also changes their residual cost, which forms the pboperty tax base. Therefore, an increase in depreciation deductions can lead to a reduction of the enterprise's fiscal payments and could, consequently, be used to attract investment.

According to Article 258 of TC RF, depreciable property is divided into ten groups, depending on their useful lifetime. In accordance with article 259 of TC RF, depreciation is calculated with either the linear (straight line) or the non-linear (formerly called declining balance) method. The depreciation is accumulated according to the depreciation rate which is an inverse to the asset's useful lifetime. With linear method, the amount of the accumulated depreciation (per month) is equal to the product of the asset 's *acquisition cost* and the depreciation rate. With the non-linear method, the amount of accumulated depreciation (per month) is defined as the product of the asset's *residual cost* and the rate of depreciation (which is twice as high as the corresponding rate for the linear method).

Article 284 of TC RF establishes the new profit tax rate to be 24% (instead of 35% under the former law). Of this, 7.5% are included in the federal budget, 14.5% — in the regional budgets and 2% — in the local budgets. From our point of view, there is a certain similarity with the former tax holidays mechanism, since regional authorities are permitted to reduce the regional profit tax rate for specific categories of taxpayers, but not more than by 4%.

The amounts of depreciation accrued and property tax paid depend, to a considerable degree, on the enterprise's accounting policy. According to the accounting rules that apply to capital funds (PBU 6/01) adopted in 2001, commercial organizations are authorized to revaluate their assets at the current market (replacement) cost. Thus, the residual replacement cost (net of wear and tear) is determined as the product of the overall replacement cost after revaluation and the ratio of its residual cost before revaluation to the balance cost before revaluation according to the accounting data. So, the impact of the revaluation mechanism can be relatively complex. On the one hand, it can lead to an increase in the depreciation deduction and thus to a decrease in the tax base; on the other hand, it can increase the value of the residual cost, i.e. it can increase the property tax.

The methodological basis of our study consists in two models. The first model describes behavior of an investor in the Russian fiscal environment, taking into account factors of risk and uncertainty. The second model presents selection of a depreciation mechanism providing for the maximum fiscal revenues into the regional budget from created enterprises.

Let us describe now the above-mentioned models at the qualitative level.

Consider an investment project that assumes the creation of a new enterprise (firm) in a certain region. The technological description of the investment project can be represented as a temporal sequence of expenditures and outputs in physical units. Prices for resources and outputs products fluctuate stochastically depending on the situation on the market. Thus, by observing the prices of input and output production and knowing the technological description of the project, one can calculate, for any moment of time, income and production costs before the creation of the firm. The creation requires a certain amount of various resources in physical units. Since market prices of the required resources change randomly, the monetary amount of necessary investment can be considered as a stochastic process. We shall assume that these investments are instantaneous and irreversible, which means that the firm starts its production right after the investment has been made, and the investment cannot be withdrawn and re-allocated for other purposes.

According to these assumptions, cash flows which are connected to profits and costs of production appear at the moment of investment. In accordance with Article 253 of TC RF, the above costs are subdivided into material costs, payroll, depreciation and other costs. According to Article 264 of TC RF, payments to social funds (unified social tax) and enterprise property tax, whose tax base depends on the depreciation and accounting policy of the project, will be referred as other costs. The above incomes and costs represent the base for calculation of the profit tax which consecutively depends on the project depreciation policy. Thus, all the firm's cash flows, defined from the moment of investment, are random processes since the market prices are stochastic.

The behavior of a potential investor is assumed to be rational in the sense that he can either make a decision to invest in the project or delay it for some time until the situation becomes more favorable. When making this decision, the investor takes into account, apart from cash flows, various potential adverse events, which may occur after the firm's creation such as the partial loss property and/or profit[1].

Thus, at any given moment, the investor can calculate the net present value of the firm, given current market prices and a forecast of future cash flows. Investor chooses the optimal moment for investment in order to maximize NPV of the project, basing on the information about the market prices observable at any moment of time. Such rule determines the investor's behavior.

The main problem that arises here and requires investigation at the model level is the influence of the tax mechanism on the investor behavior under the conditions of risk and uncertainty. Special attention will be paid to the mechanism of depreciation. For example, how does accelerated depreciation affect the investor behavior under the condition of simultaneous usage of tax holidays? This question is especially pertinent for newly created enterprises since tax holidays represent the principal fiscal exemption in most countries.

With tax reform still being implemented, the study of some innovations in the tax legislation is of great interest. First of all, it is interesting to compare the new system of corporate income taxation with the former system at model level. In other words, that is to compare the former system with tax holidays for newly created enterprises and the mechanism of accelerated depreciation to the new system without these exemptions, but with a lower profit tax rate.

Numerous economists (see, e.g., the monograph *Problems of Tax System in Russia ...*, 2000) propose to introduce the real estate tax. Similar law exists in the majority of countries with developed market economies and, as believed by the authors of the above-mentioned monograph, it should substitute for the three laws existing in the Russian tax system: the property tax charged to enterprises, the property tax charged to the physical persons and the land tax. The draft law benefits from substantial support from the bodies of legislative and executive authority. The Federal law "On the experiment with taxation of the real estate in Great Novgorod and Tver cities in 1997-2000", which was accepted in 20.06.97, shows the importance attached to this project by the government[2]. The model developed in the present paper is amended in order to analyze the influence which the substitution of the real estate tax for the property and land taxes will have on the investor behavior.

The following questions reflected in the paper cover a problem of risk compensation through the tax mechanisms. To explain this, we shall consider the

[1] A sequence of such losses we will referred to as risk process

[2] This experiment was extended to the year 2003 by a decision of the State Duma Law

following hypothetical scheme. Let's assume that the investor, acting according to the above model, faces a dilemma: whether to invest in an economy with a significant risk but with benefits from tax incentives, or in an economy that has neither the risk nor the incentives. The question of risk compensation can thus be formulated as follows. Are there, in an economy facing great risks, such tax incentives for which the investor's NPV equals or exceeds the same amount for similar projects in an economy without risk? In the present paper two tax mechanisms will be considered as such incentives: change in profit tax rate and depreciation policy.

Let us return to the problem of investment incentives with the help of fiscal stimulus. Let us consider the problem from a regional point of view. Suppose that the region is interested in the completion of certain investment projects (creation of new enterprises). The absence of such enterprises can be hazardous for the region which is forced to find elsewhere what the enterprise can produce locally and has to carry the burden of unemployment. The implantation of enterprise can increase tax revenues in the regional budget. In the 2001, the Law on Budget, corporate profit tax (for its regional part), personal income tax and property tax were maintained in the regional budget[3].

Schematically, before the firm is created (before the moment of investment), there is a flow of losses in the regional budget due to absence of the firm. And after the investment has been made, a tax flow emerges into the regional budget, connected with the firm's activity.

Let us suppose that the region owns some mechanisms (of fiscal nature), with the help of which it can influence the investor's arrival. Among these mechanisms can be quoted: the decrease of tax base by the choice of corresponding depreciation policy, tax holidays and tax credits. Let us underline that such exemptions are assigned to a *project* and not to a *concrete investor* (which could be directly linked to corruption). The information on such a project (with corresponding exemptions) is presented for potential investors.

Until recently the mechanism of accelerated depreciation for the active part of assets was used as one of the incentives tools for attraction of investment in the Russian regions. The calculation method for accelerated depreciation was linear. For this method, the "acceleration rate" could not exceed 2. A further increase of this rate was still possible but had to be approved by the local fiscal authorities. Even if the mechanism of accelerated depreciation played a great role in the improvement of investment climate, it is not stipulated anymore (in a way presented above) in the new legislation (Chapter 25 of TC RF).

[3] Local taxes will be considered as federal in the context of this paper

Tax holidays was another regional mechanism of investments stimulation for new enterprises. The new enterprise was exempted from the profit tax during the payback period, but not exceeding a period of three years. Tax holidays are also not stipulated any more in the new TC RF.

However, the study of possible above mentioned fiscal mechanisms for an investment incentive purpose at a model level is, in our opinion, not only of theoretical interest but also of great practical one.

The present report, which consists of five chapters and mathematical appendix, is structured as follows.

Chapter 2 contains a review of the literature directly related to the subject of our study.

Chapter 3 considers the general model of the investor's behavior in Russian tax environment under risk and uncertainty. The model includes the description of the cash flows structure in continuous time, main hypotheses about the investor's behavior, reduction of the problem of finding the optimal moment for investment to an optimal stopping problem for stochastic process.

Chapter 4 is devoted to the investigation of the investor's behavior model and considers the main assumptions concerning stochastic dynamics of the firm's cash flows, risk process and mechanism of assets revaluation. These assumptions have provided us with the explicit formulas for investor's NPV and tax payments into the federal and regional budgets. The main result of this chapter is the comprehensive decision of the investor's problem (Section 4.3), obtained as the rule which determines the optimal moment for investment. Dependence of this rule on the principal exogenous variables (tax rates, depreciation policy, tax holidays, parameters of the investment project, discount rate, risk and uncertainty factors) is described.

On the basis of these results the authors obtain the explicit formulas that allow to derive expected NPV of the investor as well as expected tax payments into the federal and regional budgets given the optimal behavior of the investor.

The final section of this chapter is dedicated to the study of mutual influence of tax holidays and accelerated depreciation on investor's behavior. The conditions under which the mutual usage of both mechanisms implies a decrease in the investment activity (later investment) are presented.

Chapter 5 proposes an optimization approach to the problem of fiscal stimulation. This approach allows to obtain explicitly a depreciation policy which provides maximum tax revenues in the regional budget. The chapter also analyzes the problem taking into account the restrictions upon the choice of optimal depreciation.

Chapter 6 presents the different capabilities of the models previously constructed in preceding chapters for the analysis of the Russian fiscal environment. Thus, a comparative analysis of the former system of corporate taxation and the new one is presented in Section 6.1. In Section 6.2, the replacement of property tax by the real estate tax (the model of Novgorod–Tver experiment) at a modeling level is studied. Section 6.3 is devoted to the problem of risk compensation by means of profit tax rate reduction and changes of tax base through the depreciation mechanism. At this stage we answer the following question: whether exist such zones of risk (for each mechanism) which cannot be compensated by any changes in these mechanisms (in tax rate or in depreciation policy). In Section 6.4 the "converting relation" between the depreciation rates for the linear and non-linear methods was derived which provides for identical influence of depreciation policy on optimal investment rule and related indicators. The final part of this chapter studies two schemes of losses carry forward in tax payments. One of them is commonly used under the provision of Article 283 of TC RF. The modeling of this scheme results in insuperable difficulties in obtaining the explicit formulas. The second is an approximate scheme. It is used in the model and allows us to obtain explicit formulas. Then we evaluate the rate of approximation of the first scheme by the second one according to the criterion of the expected net present value of the firm.

Chapter 7 (Mathematical appendix) is devoted to the description of a new approach to solving an optimal stopping problem for multidimensional diffusion processes. This approach is based on an intimate connection between the boundary problem for diffusion processes and the Dirichlet problem for the partial differential equations of an elliptic type. The solution of Dirichlet problem is considered as a functional of the continuation region. The optimization of this functional on the set of all available continuation regions will be carried out by variational methods. The approach described is applied to the solving of an optimal stopping problem for a two-dimensional geometric Brownian motion with objective functional, which is an expectation of homogeneous (of any non-negative degree) function of the process at the stopping moment. The problem of finding the optimal moment for investment is reduced to a problem of such a type.

2. THE REVIEW OF THE LITERATURE

There is a large quantity of publications devoted to the stimulation of investment using fiscal methods. For this reason, we will focus our attention solely on studies which are directly related to the subject of our research. Particularly, we won't refer to papers that study the influence of aggregated tax incentives on macroeconomic variables of investment activity (detailed empirical analysis of efficiency of the aggregated sub-federal tax incentives for attraction of investments was carried out in EERC framework by Kolomak, 2000) or tax competition of regions to attract investments (a survey of this subjects can be found, for example, in Inman and Rubinfeld, 1996).

Studies that deal with the influence of the tax exemptions on firms activity are much closer to our question though, in general, they consider an already established firm. So, Aukutsionek and Batyaeva (2001) conducted a study of the influence of taxation on the financial and economic state of an enterprise. Their study is based upon monitoring data of Russian enterprises within the Russian Economic Barometer program.

A considerable number of papers deals with the problem of choice of depreciation policy and its influence on the activity of the enterprises. Although accounting documents define strict rules for a depreciation policy, in many cases there remains a certain degree of freedom in the choice of both rate and method of depreciation. In papers by (Roemmich *et al.*, 1978; Berg *et al.*, 2001) minimization of the present tax payment values by choosing a particular depreciation policy has been studied. This choice may be significantly affected by the stochastic nature of future cash flows. In this respect, there are remarkable papers by Berg and Moore (1989); Berg *et al.* (1996), in which the present values of future tax payments of the firm (within the progressive tax scale) are compared for two methods of depreciation, straight-line (in equal parts) and "accelerated" (in non-equal parts, decreasing in time). Although the "accelerated depreciation" might seem a priori the most favorable (due to discount effect), the optimal choice of depreciation method may in fact depend on the degree of uncertainty of cash flows, discount rate, fiscal regulations, and the possibility of transferring the losses onto other periods of time. Wakeman (1980) has shown that if the tax rate is flat and if taxable income is non-negative in all periods for all available depreciation methods, the accelerated depreciation method is most preferable for tax purposes. This is a consequence of the fact that a more accelerated method typically shifts taxable income to later periods, and when future money is discounted, paying taxes later is preferable to paying them now. Wilhouwer *et al.* (2001) consider a similar situation but allow for uncertainty in future cash flows as well as a progressive tax structure. It is shown that a less accelerated method can then be optimal. Sansing (1998), Wilhouwer *et al.* (1999) studied the influence of tax depreciation and technical depreciation on investment activity.

The starting point of this paper's model (the investor's behavior under uncertainty), is the McDonald- Siegel model (McDonald and Siegel, 1986), which is the basis for the well-known real options theory. Several monographs, for example, Dixit and Pindyck (1994), Trigeorgis (1996) are devoted to this subject. This model studies the behavior of the investor, whose present value after making investment in the project is described by a stochastic process (usually a geometric Brownian motion), and investments are considered to be irreversible. The goal is to find the optimal (according to NPV criterion) moment for investment, which is considered as the optimal stopping time in the observation of present value.

The real options theory is represented as the "handy" and adequate tool for modeling process of creation of new firm. However there are only a few studies dealing with the influence of tax system on investor's behavior within this framework. MacKie-Mason (1990) studied the interaction of uncertainty and non-linear tax boundaries for projects in the mining industry. It was shown, that a certain combination of profits uncertainty and non-linear profit tax can result in unexpected effects (growth of tax rate can encourage investment, and the increase of deductions from the total profit can discourage investment). Forsfalt (1999) on the basis of the "real options" approach to the creation of small firms compared various tax systems of Nordic style. In particular, it was shown that the comprehensive income tax system (where all incomes are taxed regardless of their source) yields a higher threshold for creating a firm than the "dual" system (where incomes are separated with respect to the type of income).

With reference to the Russian system of profit taxation the investment waiting model, which is a development of McDonald-Siegel model, was proposed by Arkin and Slastnikov (1997). The extension of this model by the introduction of a lag period between the moment of investment and production was studied by Arkina and Slastnikov (1999). Under the assumption of time-independent amount of investment the optimal rule for investment was obtained in an analytical form depending on the parameters of the model (characteristics of the investment project, discount rate, profit taxation system). Explicit relationships of certain indices of investment activity such as the expected time of investment waiting, the probability of investing in the project, the expected tax payments into the federal and regional budgets, which all depend on the above mentioned parameters were analyzed (Arkin *et al.*, 1999b).

Recently, participants of the project started to incorporate the depreciation method into the investment waiting model. In Arkin and Slastnikov (2000a), formulas for the optimal rule for investment with explicit accounting for the depreciation policy and tax holidays were obtained. The "optimal" depreciation policy, which maximizes the present tax payments from the investment project has been found in Arkin and Slastnikov (2000b). Numeric calculations of the optimal depreciation rates for the current methods with additional restrictions have been carried out using adjusted real data.

The paper by Arkin and Slastnikov (2001) is devoted to the analysis of new effects arising under combined use of accelerated depreciation and tax holidays. It has been shown that under certain conditions an increase of tax holidays leads to later investment moment, thus decreasing investment activity. A similar effect has been discovered by Abel (1980), Mintz (1990) using a different model. On the other hand, introducing accelerated depreciation does not always stimulate the investor in the presence of tax holidays.

Investigation of the model of regional stimulation on the basis of the investment waiting model has been started by Arkin *et al.* (1999a). Tax holidays, exempting the investor from paying the regional part of the corporate profit tax, were considered as a stimulation mechanism. The principle of determination of the "optimal" tax holidays, when the expected present tax payments from the firm into the regional budget is maximized, has been introduced. The area of parameters of the model was specified, in which an increase of tax holidays is favorable both to the investor and to the regional and federal authorities, since it provides an increase of their incomes (area of mutual benefits). In addition, we have compared the proposed "optimal" principle of determination of tax holidays with the methods currently in use in Russian regions, namely the payback period principle and fixed (within a given region) tax holidays, usually 3-5 years.

It is worth noting that all of the above-mentioned papers assume the profit of the firm to be positive, consider only the profit taxation, and consider the amount of investments required to create the firm to be time-independent. The latter assumption prohibits the inclusion in the model of such an important element of depreciation policy as revaluation of assets. Additionally the depreciation of assets is directly connected with their residual cost, and therefore with the corporate property tax, which has not been considered at all in the papers cited above. The statements specified above considerably restrict the field of problems under consideration and the application of obtained results. The construction of a model of investor's behavior, freed of statements specified above, is one of the problems of the present work.

3. DESCRIPTION OF THE BASIC MODEL

In this Chapter a general model of investor behavior in Russian fiscal environment under risk and uncertainty will be described. The basic result of the study of this model will be the obtaining (in Chapter 4) in an explicit (analytical) formula of the dependence of optimal investment moment (rule) on parameters of the investment project and economic environment.

As object of investment, a project for the creation of a new industrial enterprise (firm) in a certain region will be considered, producing certain goods and consuming certain resources. Investments necessary for the project (of creation and start of the new firm), are considered to be instantaneous and irreversible so that they cannot be withdrawn from the project any more and used for other purposes (sunk cost).

An important feature of considered model will be the assumption that, at any moment, the investor can either *accept* the project and start with the investment or *delay* the decision until he obtains new information about its environment (prices of the product and resources, demand etc.).

3.1. Cash flows structure

Let us suppose that investment in the project starts at moment τ.

Let gross income from the firm at time $t \geq \tau$ be X_t^τ, and production cost at time t be $C_t^\tau = Y_t^\tau + S_t^\tau + D_t^\tau + M_t^\tau$, where Y_t^τ is material cost (including cost of raw materials, etc.), S_t^τ is payroll cost, D_t^τ is depreciation charges at this moment, M_t^τ – other costs, in which are included enterprise property tax (assets tax) P_t^τ and social funds payments $\gamma_s S_t^\tau$ (at the rate γ_s), i.e. $M_t^\tau = P_t^\tau + \gamma_s S_t^\tau$[4].

Tax base for the calculation of profit tax is

$$Z_t^\tau = X_t^\tau - C_t^\tau = \pi_t^\tau - S_t^\tau - D_t^\tau - M_t^\tau \text{[5]}, \tag{3.1}$$

where $\pi_t^\tau = X_t^\tau - Y_t^\tau$ is value added[6].

Let us define the tax rates, which will be used later on.

[4] The gross income X_t^τ and the material costs Y_t^τ are considered VAT excluded

[5] In fact in case of losses, i.e. when costs exceed income this is not necessarily right. The peculiar problem of losses will be treated in Section 6

[6] Sometimes by value added cost it is meant the difference between income and material costs taking into account VAT

γ_i is the enterprise profit tax rate, consisting on the rate into the federal budget γ_i^f and the rate into the regional budget[7] γ_i^r.

γ_p is the enterprise property tax (assets tax) rate.

γ_{va} is the VAT rate, which is splitted[8] into the federal γ_{va}^f and the regional γ_{va}^r parts.

γ_s is the payroll tax rate which is, as well, splitted into the federal γ_s^f and the regional γ_s^r parts.

γ_{pi} is the personal income tax rate, splitting into the federal γ_{pi}^f and the regional γ_{pi}^r parts.

The total taxes, paid by the firm at moment t, are equal to

$$\gamma_{va}\pi_t^\tau + \gamma_i(\pi_t^\tau - S_t^\tau - D_t^\tau - M_t^\tau) + P_t^\tau + \gamma_s S_t^\tau.$$

The payments into the federal budget are

$$\gamma_{va}^f\pi_t^\tau + \gamma_i^f(\pi_t^\tau - S_t^\tau - D_t^\tau - M_t^\tau) + (\gamma_s^f + \gamma_{pi}^f)S_t^\tau, \tag{3.2}$$

and into the regional budget are

$$\gamma_{va}^r\pi_t^\tau + \gamma_i^r(\pi_t^\tau - S_t^\tau - D_t^\tau - M_t^\tau) + P_t^\tau + (\gamma_s^r + \gamma_{pi}^r)S_t^\tau. \tag{3.3}$$

(the personal income taxes from the firm's employees are included).

After-tax cash flow of the firm at the moment t is equal to

$$X_t^\tau - Y_t^\tau - S_t^\tau - M_t^\tau - \gamma_i(X_t^\tau - C_t^\tau) = (1-\gamma_i)(\pi_t^\tau - S_t^\tau - M_t^\tau) + \gamma_i D_t^\tau. \tag{3.4}$$

3.2. Assets evaluation, depreciation, enterprise property tax

In our model, the base for depreciation charges and connected taxes will be the balance cost of assets. Let I_t^τ be the balance cost of assets at the moment t provided the firm is created at the moment τ. Index τ emphasizes in the model, that the cost of assets (which practically is the same as the cost of required investments) depends on the moment of investment. Dependence of I_t^τ on current moment of time t means, that the initial cost of assets after investment can be revaluate (for example, by replacement cost) in accordance with current economic situation. Let us point out the two most important

[7] Under "regional budget" we will consider sub-federal plus local budget

[8] Coefficients of "splitting" are determined annually in the federal budget law

cases: the case $I_t^\tau = I_\tau$ corresponds to the absence of assets revaluation (after the investment), and $I_t^\tau = I_t$ can be interpreted as "continuous" revaluation (assets are permanently revaluate according to current market prices).

As it was already mentioned, in accordance with TC RF, all depreciated assets are divided into 10 groups depending on their useful lifetime (from 1 year up to 30 years and higher). Within the framework of our model we divide all assets into two aggregated parts: one of them (we will call it an "active" part) refers to machinery, tools, equipment etc. (its share in balance cost of all assets will be denoted as $\psi,\ 0 \le \psi \le 1$); and the other ("inactive" part) refers to buildings and structures, which useful lifetime is long enough.

Depreciation charges at the moment of time t for the project started at the moment τ will be:

$$D_t^\tau = \psi I_t^\tau a_{t-\tau} + (1-\psi) I_t^\tau b_{t-\tau}, \quad t \ge \tau \tag{3.5}$$

where I_t^τ is the balance cost of all assets, ψ is the share of active part, $(a_t, t \ge 0)$, $(b_t,\ t \ge 0)$ are the "densities" of depreciation of active and inactive parts of assets such that:

$$a_t,\ b_t \ge 0, \quad \int_0^\infty a_t\, dt = \int_0^\infty b_t\, dt = 1.$$

Such scheme covers various depreciation methods (more exactly, their variants in continuous time), accepted by modern tax laws. The TC RF allows two methods of this type. With the linear method (SL) the depreciation is charged by equal shares during all useful lifetime. The corresponding density is:

$$a_t = \begin{cases} \lambda, & \text{if } 0 \le t \le L \\ 0, & \text{if } t > L, \end{cases} \tag{3.6}$$

where L is a duration of the useful lifetime, and $\lambda = 1/L$.

According to the non-linear method (DB) the charged depreciation at each moment of time is defined as a product of the fixed rate by the residual cost of asset. One can show, that this method can be described by exponential density with rate $\eta > 0$

$$a_t = \eta e^{-\eta t}. \tag{3.7}$$

The depreciation influences tax bases of two taxes: the profit tax (see (3.1)) and the property tax (assets tax), the later will be considered in detail. Since tax base for this case is the residual cost of assets we can present assets tax within the framework of our model as

$$P_t^\tau = \gamma_p I_t^\tau [1 - \psi \widehat{a}_{t-\tau} - (1-\psi)\widehat{b}_{t-\tau}], \quad t \geq \tau \tag{3.8}$$

where

$$\widehat{a}_\theta = \int\limits_0^\theta a_s\, ds, \quad \widehat{b}_t = \int\limits_0^\theta b_s\, ds \tag{3.9}$$

are accumulated shares of depreciation (at balance cost) for, respectively, active and inactive part of assets during the period θ after investment.

Let us describe briefly the experiment on the real estate taxation, which began in 1997 in Tver and Novgorod and was prolonged for three years from 2000. The main idea of this experiment is to replace several property taxes such as enterprise property tax, physical persons property tax and land tax by the single real estate tax. The base of this tax is the market price of the real estate. Land is also an object of taxation, that is why cost of any real estate object includes a land component.

In our model real estate is linked to the inactive part of assets (plus land where they are located). Therefore, within the framework of Novgorod–Tver experiment the enterprise property tax can be approximately replaced by

$$\tilde{P}_t^\tau \approx \tilde{\gamma}_p (1-\psi) I_t,$$

where $\tilde{\gamma}_p$ is the real estate tax rate, and $(1-\psi)I_t$ is a cost of the inactive part of assets calculated at the current market prices[9]. In other words, the tax base for the enterprise property tax is not the residual cost of assets but its full replacement cost. We come back to the discussion of this experiment at the model level in Chapter 6.

3.3. Uncertainty and risk. Investment timing

Let investment in the project start at the moment τ, and I_τ be the amount of required investment.

Since economic environment can be subject to various random factors influence (uncertainty in market prices, demand etc.), we will consider the amount of required investment $(I_t,\ t \geq 0)$ as a random process, and value added $(\pi_t^\tau,\ t \geq \tau,\ \tau \geq 0)$ will be described as a family of random processes given on some probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ with a flow of σ-fields $(\mathcal{F}_t,\ t \geq 0)$. $\mathcal{F}_t$ can be interpreted as the observable information about the system up to the moment of time t. I_t and π_t^τ are assumed to be $\mathcal{F}_t$-measurable.

[9] Land cost here is not included, because methods for its evaluation are not developed yet

Let $(\zeta_t^\tau,\ t \geq \tau,\ \tau \geq 0),\ 0 \leq \zeta_t^\tau \leq 1$ be the risk process acting on after-tax cash flow of the investor. This risk process is connected with the loss of a share of the investor's net profit at random moments of time after investing.

As we mentioned in Introduction, according to new TC RF the regional authorities can reduce (in certain limits) profit tax rate in its regional part (Article 284). Let ν be the length of the interval of time (after creation of the firm), during which lower regional profit tax rate γ_i^0 is applicable[10], and $\bar{\gamma}_i = \gamma_i^f + \gamma_i^0$ is lower profit tax rate, which is charged to the investor.

For simplicity we assume that the firm earns profit right after the investment is made. Then, according to (3.4), the present value of investor (discounted to the moment of investment) can be expressed as the following formula

$$\begin{aligned} V_\tau &= \mathbf{E}\left(\int\limits_\tau^{\tau+\nu} \zeta_t^\tau [(1-\bar{\gamma}_i)(\pi_t^\tau - S_t^\tau - M_t^\tau) + \bar{\gamma}_i D_t^\tau] e^{-\rho(t-\tau)}\, dt \right. \\ &\quad \left. + \int\limits_{\tau+\nu}^{\infty} \zeta_t^\tau [(1-\gamma_i)(\pi_t^\tau - S_t^\tau - M_t^\tau) + \gamma_i D_t^\tau] e^{-\rho(t-\tau)}\, dt \,\middle|\, \mathcal{F}_\tau \right), \end{aligned} \tag{3.10}$$

where ρ is discount rate, and $\mathbf{E}(\cdot|\mathcal{F}_\tau)$ stands for the conditional expectation provided by information about the system up to the moment τ.

The behavior of the investor is assumed to be rational in the sense that he chooses the moment of investment τ (investment rule), in order to maximize his expected net present value (NPV):

$$\mathbf{E}\,(V_\tau - I_\tau)\, e^{-\rho\tau} \to \max_\tau, \tag{3.11}$$

where the maximum is considered over all Markov moments τ.

Simultaneously to NPV, it is possible to calculate, using formulas (3.2) and (3.3), the present tax payments from the firm into the budgets. The expected present tax payments from the firm into the *federal budget*, discounted to moment τ are equal to:

$$T_\tau^f = \mathbf{E}\left(\int\limits_\tau^\infty [\gamma_{va}^f \pi_t^\tau + \gamma_i^f(\pi_t^\tau - S_t^\tau - D_t^\tau - M_t^\tau) + \tilde{\gamma}_s^f S_t^\tau] e^{-\rho(t-\tau)}\, dt \,\middle|\, \mathcal{F}_\tau \right), \tag{3.12}$$

and into the *regional budget* –

[10] This period is usually called tax holidays, though this term is absent in modern tax laws

$$T^r_\tau = \mathbf{E}\left(\int\limits_\tau^{\tau+\nu} [\gamma^r_{va}\pi^\tau_t + \gamma^0_i(\pi^\tau_t - S^\tau_t - D^\tau_t - M^\tau_t) + P^\tau_t + \tilde{\gamma}^r_s S^\tau_t] e^{-\rho(t-\tau)} dt \right.$$
$$\left. + \int\limits_{\tau+\nu}^{\infty} [\gamma^r_{va}\pi^\tau_t + \gamma^r_i(\pi^\tau_t - S^\tau_t - D^\tau_t - M^\tau_t) + P^\tau_t + \tilde{\gamma}^r_s S^\tau_t] e^{-\rho(t-\tau)} dt \,\middle|\, \mathcal{F}_\tau \right), \quad (3.13)$$

where $\tilde{\gamma}^f_s = \gamma^f_s + \gamma^f_{pi}, \quad \tilde{\gamma}^r_s = \gamma^r_s + \gamma^r_{pi}$.

Another interesting indicator is the expected present tax burden $\mathcal{B}_\tau$ for the created firm which is defined as a ratio of expected present tax payments from the firm to expected present value added, i.e.

$$\mathcal{B}_\tau = \mathbf{E}\left(\int\limits_\tau^{\infty} [\gamma_{va}\pi^\tau_t + \gamma^f_s S^\tau_t] e^{-\rho(t-\tau)}\, dt + \int\limits_\tau^{\tau+\nu} \bar{\gamma}_i Z^\tau_t e^{-\rho(t-\tau)}\, dt \right.$$
$$\left. + \int\limits_{\tau+\nu}^{\infty} \gamma_i Z^\tau_t e^{-\rho(t-\tau)}\, dt \,\middle|\, \mathcal{F}_\tau \right) \Big/ \mathbf{E}\left(\int\limits_\tau^{\infty} \pi^\tau_t e^{-\rho(t-\tau)}\, dt \,\middle|\, \mathcal{F}_\tau \right), \quad (3.14)$$

where tax base Z^τ_t is defined in (3.1).

4. SOLUTION OF THE INVESTOR PROBLEM

In this Chapter we provide a solution for the investor problem which as it turned out can be obtained in an explicit (analytical) form. On the basis of the obtained formulas we will carry out a theoretical analysis as well as numeric calculations.

4.1. Main assumptions

The amount of required investment I_t is described by geometric Brownian motion

$$I_t = I_0 + \int\limits_0^t I_s(\alpha_1\, ds + \sigma_1\, dw^1_s), \quad t \geq 0, \quad (4.1)$$

where $(w^1_t,\ t \geq 0)$ is a Wiener process, α_1 and σ_1 are real numbers ($\sigma_1 \geq 0$), and I_0 is a given initial state of the process.

The dynamics of *value added* $\pi_t^\tau,\ t \geq \tau$ is specified by a family of stochastic equations

$$\pi_t^\tau = \pi_\tau + \int_\tau^t \pi_s^\tau(\alpha_2\,ds + \sigma_2\,dw_s^2), \quad t \geq \tau, \tag{4.2}$$

where π_τ is $\mathcal{F}_\tau$-measurable random variable, $(w_t^2,\ t \geq 0)$ is a Wiener process, α_2 and σ_2 are real numbers ($\sigma_2 \geq 0$).

The pair (w_t^1, w_t^2) is two-dimensional Wiener process with correlation r, i.e. $\mathbf{E}(w_t^1 w_t^2) = rt$ for all $t \geq 0$.

We assume that at any moment τ, observing the current prices on both input and output production one can calculate $\pi_\tau = \pi_\tau^\tau$, which is the difference between incomes and material costs at the investment moment, i.e. value added at the "initial moment" of creation of firm, and, hence, can evaluate the future profits from the project before the actual creation of the firm. For these reasons we will refer to $(\pi_t,\ t \geq 0)$ as "virtual" value added of the project. Knowing the information about virtual value added of the project as well as about the amount of required investment, investor can calculate (by formula (3.10)) an expected net present value of the project provided the investment would be made at that (considered) moment. Thus, we guess that investor makes a decision about investing in the project on the ground of observations on two-dimensional stochastic process $((\pi_t, I_t),\ t \geq 0)$. Therefore, without loss of generality, we consider that σ-field $\mathcal{F}_t$ is generated by the values of this two-dimensional process up to the moment t, i.e. $\mathcal{F}_t = \sigma(\pi_s, I_s;\ 0 \leq s \leq t)$.

We suppose that the process of virtual value added $(\pi_t,\ t \geq 0)$ is subject to the stochastic equation

$$\pi_t = \pi_0 + \int_0^t \pi_s(\alpha_2\,ds + \sigma_2\,dw_s^2), \quad t \geq 0, \tag{4.3}$$

with given initial state π_0.

As one can see from the above formulas, a process of geometric Brownian motion (that is non-negative process) is the basis for the description of dynamics of both value added and amount of required investments. Such an hypothesis is typical for many financial models including known real options theory (Dixit and Pindyck, 1994; Trigeorgis, 1996), and follows from a general assumptions about stochastic processes like independence, homogeneity, continuity (see, for example, Arkin *et al.*, 1999a). The parameters of the geometric Brownian motion have a natural economic interpretation: the drift coefficient (at dt) is an expected instantaneous rate of process changes; and the diffusion coefficient (at dw_t^i) is an instantaneous variance of process changes (volatility). Thus, this process can be viewed as an approximation for the relevant actual processes.

As *risk process* $(\zeta_t^\tau,\ t \geq \tau,\ \tau \geq 0)$ we will consider the random processes (with jumps) of the following type

$$\zeta_t^\tau = \prod_{j=1}^{N_t^\tau} (1 - \xi_j),$$

where $0 \leq \xi_j \leq 1$ are shares of losses in after-tax investor's profit, and N_t^τ is a number of adverse events (moments of losses) appeared at the interval $[\tau, t]$.

Shares of losses $\xi_j,\ j = 1, 2, \ldots$ are assumed to be identically distributed and independent (mutually and on Wiener process $(w_t^2,\ t \geq 0)$) random variables. The intervals between the adverse events are independent and exponentially distributed (with parameter ϵ) random variables. These assumptions imply that a number of adverse events $N_t^\tau = N_{t-\tau},\ (N_s,\ s \geq 0)$ is Poison process with parameter ϵ.

Note, that the parameters of a risk process, namely, the intensity of appearing of the adverse events ϵ and the average share of losses $\mathbf{E}\xi_j$ can, in general, depend on parameters of the project.

The assumptions formulated above reflect such properties of a risk process as unpredictability of losses both in time and in amount.

As for *assets revaluation*, we accept the hypothesis that the forecasted balance cost of assets I_t^τ will "follow" the dynamics of investments cost I_t. Namely, we assume that

$$\mathbf{E}\left(I_t^\tau \mid \mathcal{F}_\tau\right) = I_\tau e^{\theta \alpha_1 (t-\tau)}, \quad 0 \leq \theta \leq 1, \tag{4.4}$$

where the parameter θ characterizes revaluation mechanism. As follows from (4.1), the case $\theta = 0$ corresponds to no-revaluation of assets, and the case $\theta = 1$ corresponds to "continuous" revaluation of assets according their replacement value I_t. If one thinks that revaluation occurs at random times (subjected to Poison distribution), then parameter θ can be associated with the intensity of this underlying process.

Share of active part ψ of depreciable assets is fixed.

The payroll fund S_t^τ is supposed to be proportional to the value added π_t^τ, i.e.

$$S_t^\tau = \tilde{\mu} \pi_t^\tau,$$

where $\tilde{\mu}$ is a given constant. Though such a hypothesis is rather disputed it is in accord with the principle of a dependence between wages and production activity. In some sense it is technical because its rejection leads to multi-dimensional diffusion processes (with dimension greater than 2) that in turn makes it impossible to obtain explicit (analytical) formulas.

Taking into account this assumption the taxable profit (tax base) (3.1) can be written as $X_t^\tau - C_t^\tau = \pi_t^\tau - S_t^\tau - D_t^\tau - P_t^\tau - \gamma_s S_t^\tau = \pi_t^\tau(1-\mu) - D_t^\tau - P_t^\tau$, where $\mu = (1+\gamma_s)\tilde{\mu}$. In order for this tax base not to be negative at all times we have to require $\mu < 1$, or $(1+\gamma_s)\tilde{\mu} < 1$.

4.2. Derivation of investor's present value and present tax payments

Now we can write explicit formulas for the investor's present value and for present tax payments into the budgets.

We will denote a conditional expectation provided by $\mathcal{F}_\tau$ as $\mathbf{E}_\tau$.

Begin with the derivation of $\mathbf{E}_\tau \zeta_t^\tau, \quad t > \tau$. Taking into account the independence of both the variables ξ_j and Poison process N_t^τ on Wiener process (w_t^2), we have

$$\begin{aligned}
\mathbf{E}_\tau \zeta_t^\tau &= \mathbf{E} \prod_{j=1}^{N_t^\tau} (1-\xi_j) = \sum_{n\geq 0} \mathbf{P}\{N_t^\tau = n\} \mathbf{E} \prod_{j=1}^{n} (1-\xi_j) \\
&= \sum_{n\geq 0} \frac{\epsilon^n (t-\tau)^n}{n!} e^{-\epsilon(t-\tau)} (1-q)^n = e^{-\delta(t-\tau)},
\end{aligned}$$

where $q = \mathbf{E}\xi_j,\ \delta = \epsilon q$ is the average share of losses per unit of time.

The known properties of geometric Brownian motion and (4.4) imply:

$$\begin{aligned}
\mathbf{E}_\tau \pi_t^\tau &= \pi_\tau e^{\alpha_2(t-\tau)}, \\
\mathbf{E}_\tau I_t^\tau &= I_\tau e^{\theta\alpha_1(t-\tau)}, \\
\mathbf{E}_\tau P_t^\tau &= \gamma_p I_\tau [1 - \psi \widehat{a}_{t-\tau} - (1-\psi)\widehat{b}_{t-\tau}] e^{\theta\alpha_1(t-\tau)}, \\
\mathbf{E}_\tau D_t^\tau &= I_\tau [\psi a_{t-\tau} + (1-\psi) b_{t-\tau}] e^{\theta\alpha_1(t-\tau)},
\end{aligned}$$

Let us accept the following designations:

$$\tilde{\rho} = \rho + \delta - \theta\alpha_1, \quad A_t = \int_t^\infty a_s e^{-\tilde{\rho}s}\, ds, \quad B_t = \int_t^\infty b_s e^{-\tilde{\rho}s}\, ds, \tag{4.5}$$

$$\widehat{A}_t = \int_t^\infty (e^{-\tilde{\rho}t} - e^{-\tilde{\rho}s}) a_s\, ds, \quad \widehat{B}_t = \int_t^\infty (e^{-\tilde{\rho}t} - e^{-\tilde{\rho}s}) b_s\, ds. \tag{4.6}$$

Using (3.9), one can obtain:

$$\int_0^\infty \widehat{a}_t e^{-\tilde{\rho}t}\, dt = \int_0^\infty a_s \int_s^\infty e^{-\tilde{\rho}t}\, dt\, ds = \frac{1}{\tilde{\rho}} A_0, \quad \int_0^\infty \widehat{b}_t e^{-\tilde{\rho}t}\, dt = \frac{1}{\tilde{\rho}} B_0,$$

$$\int_\nu^\infty \widehat{a}_t e^{-\tilde{\rho}t}\, dt = \int_0^\infty a_s \int_\nu^\infty e^{-\tilde{\rho}t}\, dt\, ds - \int_\nu^\infty a_s \int_\nu^s e^{-\tilde{\rho}t}\, dt\, ds = \frac{1}{\tilde{\rho}}[e^{-\tilde{\rho}\nu} - \widehat{A}_\nu],$$

$$\int_\nu^\infty \widehat{b}_t e^{-\tilde{\rho}t}\, dt = \frac{1}{\tilde{\rho}}[e^{-\tilde{\rho}\nu} - \widehat{B}_\nu],$$

$$\int_\tau^{\tau+\nu} \mathbf{E}_\tau P_t^\tau e^{-(\rho+\delta)(t-\tau)}\, dt = \frac{\gamma_p}{\tilde{\rho}} I_\tau [1 - \psi(A_0 + \widehat{A}_\nu) - (1-\psi)(B_0 + \widehat{B}_\nu)],$$

$$\int_{\tau+\nu}^\infty \mathbf{E}_\tau P_t^\tau e^{-(\rho+\delta)(t-\tau)}\, dt = \frac{\gamma_p}{\tilde{\rho}} I_\tau [\psi \widehat{A}_\nu + (1-\psi)\widehat{B}_\nu],$$

$$\int_\tau^{\tau+\nu} \mathbf{E}_\tau D_t^\tau e^{-(\rho+\delta)(t-\tau)}\, dt = I_\tau [\psi(A_0 - \widehat{A}_\nu) + (1-\psi)(B_0 - \widehat{B}_\nu)],$$

$$\int_{\tau+\nu}^\infty \mathbf{E}_\tau D_t^\tau e^{-(\rho+\delta)(t-\tau)}\, dt = I_\tau [\psi A_\nu + (1-\psi) B_\nu].$$

Therefore,

$$\begin{aligned}
V_\tau = &\int_\tau^{\tau+\nu} \{(1-\bar{\gamma}_i)[(1-\mu)\mathbf{E}_\tau \pi_t^\tau - \mathbf{E}_\tau P_t^\tau] + \bar{\gamma}_i \mathbf{E}_\tau D_t^\tau\} e^{-(\rho+\delta)(t-\tau)}\, dt \\
&+ \int_{\tau+\nu}^\infty \{(1-\gamma_i)[(1-\mu)\mathbf{E}_\tau \pi_t^\tau - \mathbf{E}_\tau P_t^\tau] + \gamma_i \mathbf{E}_\tau D_t^\tau\} e^{-(\rho+\delta)(t-\tau)}\, dt \\
&= \frac{(1-\mu)(1-\widehat{\gamma}_i)}{\rho+\delta-\alpha_2} \pi_\tau + I_\tau [\psi(\bar{\gamma}_i A_0 + \Delta\gamma_i A_\nu) + (1-\psi)(\bar{\gamma}_i B_0 + \Delta\gamma_i B_\nu)]
\end{aligned}$$

$$-\frac{\gamma_p}{\rho+\delta-\theta\alpha_1} I_\tau\{1-\bar{\gamma}_i-\psi[(1-\bar{\gamma}_i)A_0+\Delta\gamma_i\widehat{A}_\nu]-(1-\psi)[(1-\bar{\gamma}_i)B_0+\Delta\gamma_i\widehat{B}_\nu]\}$$
$$=\frac{(1-\mu)(1-\widehat{\gamma}_i)}{\rho+\delta-\alpha_2}\pi_\tau + I_\tau\left(H_1-\frac{\gamma_p}{\rho+\delta-\theta\alpha_1}H_2\right), \qquad (4.7)$$

where

$$\Delta\gamma_i = \gamma_i^r - \gamma_i^0, \quad \widehat{\gamma}_i = \bar{\gamma}_i + \Delta\gamma_i e^{-(\rho+\delta-\alpha_2)\nu},$$
$$H_1 = \psi(\bar{\gamma}_i A_0 + \Delta\gamma_i A_\nu) + (1-\psi)(\bar{\gamma}_i B_0 + \Delta\gamma_i B_\nu),$$
$$H_2 = 1-\bar{\gamma}_i-\psi[(1-\bar{\gamma}_i)A_0+\Delta\gamma_i\widehat{A}_\nu]-(1-\psi)[(1-\bar{\gamma}_i)B_0+\Delta\gamma_i\widehat{B}_\nu].$$

As one can see from formula (4.7) for investor's present value, the parameters of risk process ζ_t^τ are included in formulas only as addition of the average share of losses per unit of time δ to discount rate ("risk premium").

Analogously, one can obtain the following expressions for the present tax payments (3.12) and (3.13) into the federal and the regional budgets, as well as the present tax burden (3.14):

$$T_\tau^f = \frac{\gamma_{va}^f + \gamma_i^f(1-\mu) + \tilde{\gamma}_s^f\tilde{\mu}}{\rho-\alpha_2}\pi_\tau - \gamma_i^f I_\tau\left\{\frac{\gamma_p}{\rho-\theta\alpha_1} + \left(1-\frac{\gamma_p}{\rho-\theta\alpha_1}\right)\right.$$
$$\left.\times\left[\psi A_0^0 + (1-\psi)B_0^0\right]\right\},$$
$$T_\tau^r = \frac{\gamma_{va}^r + (\gamma_i^0 + \Delta\gamma_i e^{-(\rho-\alpha_2)\nu})(1-\mu) + \tilde{\gamma}_s^r\tilde{\mu}}{\rho-\alpha_2}\pi_\tau - I_\tau\left(H_1^0 - \frac{\gamma_p}{\rho-\theta\alpha_1}H_2^0\right),$$
$$\mathcal{B}_\tau = \gamma_{va} + \gamma_i(1-\mu) + \tilde{\gamma}_s\tilde{\mu} - \frac{I_\tau(\rho-\alpha_2)}{\pi_\tau}\left[\frac{\gamma_p}{\rho-\theta\alpha_1}(1-K_0) - \gamma_i K_0\right],$$

where

$$H_1^0 = \psi(\gamma_i^0 A_0^0 + \Delta\gamma_i A_\nu^0) + (1-\psi)(\gamma_i^0 B_0^0 + \Delta\gamma_i B_\nu^0),$$
$$H_2^0 = 1-\gamma_i^0-\psi[(1-\gamma_i^0)A_0^0+\Delta\gamma_i\widehat{A}_\nu^0]-(1-\psi)[(1-\gamma_i^0)B_0^0+\Delta\gamma_i\widehat{B}_\nu^0],$$
$$K_0 = \psi A_0^0 + (1-\psi)B_0^0,$$

and values $A_0^0,\ B_0^0,\ A_\nu^0,\ B_\nu^0$ are determined similar to (4.5)–(4.6) with the replacement δ by 0.

Note, that if tax holidays are absent, the above formulas are rather simplified. So, the investor's present value will be

$$V_\tau = \frac{(1-\mu)(1-\gamma_i)}{\rho+\delta-\alpha_2}\pi_\tau + I_\tau\left[\left(\gamma_i + \frac{\gamma_p(1-\gamma_i)}{\rho+\delta-\theta\alpha_1}\right)K - \frac{\gamma_p(1-\gamma_i)}{\rho+\delta-\theta\alpha_1}\right], \qquad (4.8)$$

where $K = \psi A_0 + (1-\psi)B_0$.

It is worth noting that under the absence of tax holidays the present value of the investor and, therefore, his behavior depends only on the present depreciation charges for the all period after investing, i.e. on the expression of the type $\int\limits_0^\infty a_t e^{-\rho t}\,dt$. We will discuss this fact in details in Chapter 6.

4.3. Optimal moment for investment

The problem faced by the investor is an optimal stopping problem for two-dimensional stochastic process.

Let β be a positive root of the quadratic equation

$$\frac{1}{2}\tilde{\sigma}^2\beta(\beta-1)+(\alpha_2-\alpha_1)\beta-(\rho-\alpha_1)=0, \tag{4.9}$$

$\tilde{\sigma}^2=\sigma_1^2-2r\sigma_1\sigma_2+\sigma_2^2$ be a “total” volatility of investment project. It is easy to see that $\beta>1$ whenever $\rho>\alpha_2$. If $\tilde{\sigma}>0$, then

$$\beta=\frac{1}{2}-\frac{\alpha_2-\alpha_1}{\tilde{\sigma}^2}+\sqrt{\left(\frac{\alpha_2-\alpha_1}{\tilde{\sigma}^2}-\frac{1}{2}\right)^2+\frac{2(\rho-\alpha_1)}{\tilde{\sigma}^2}}.$$

The following theorem characterizes completely an optimal rule for investing.

Theorem 1. *Let the amount of required investments I_t be described by the process (4.1), and values added π_t by the relation (4.3). Assume that $\tilde{\sigma}>0$ and the following conditions are satisfied:*

$$\alpha_2-\frac{1}{2}\sigma_2^2\ge\alpha_1-\frac{1}{2}\sigma_1^2,\quad \rho>\max(\alpha_1,\alpha_2).$$

Then the optimal moment for investment is

$$\tau^*=\min\{t\ge 0:\ \pi_t\ge p^*I_t\}, \tag{4.10}$$

where $p^*=\left(1-H_1+\dfrac{\gamma_p}{\rho+\delta-\theta\alpha_1}H_2\right)\cdot\dfrac{\rho+\delta-\alpha_2}{(1-\mu)(1-\widehat{\gamma}_i)}\cdot\dfrac{\beta}{\beta-1},$ $H_1,\ H_2,$ *and* $\widehat{\gamma}_i$ *are defined in (4.7).*

Let us formulate this result for the case when tax holidays are absent.

Corollary. *If $\nu = 0$, then the optimal moment for the investment is*

$$p^* = \left[1 + \left(\frac{\gamma_i}{1-\gamma_i} + \frac{\gamma_p}{\rho+\delta-\theta\alpha_1}\right)(1-K)\right] \cdot \frac{\rho+\delta-\alpha_2}{1-\mu} \cdot \frac{\beta}{\beta-1},$$

where K is defined in (4.8).

This theorem shows that the optimal moment for investment begins when the ratio of virtual value added to the amount of required investment achieves a critical level p^*.

One can look at equality (4.10) from another point of view.

As one can see from formula (4.7) for the investor's present value V_τ, the optimal moment for the investment coincides with the moment when profitability index of the initial investment V_τ / I_τ achieves threshold level

$$\frac{\beta}{\beta-1}\left(1 - H_1 + \frac{\gamma_p}{\rho+\delta-\theta\alpha_1} H_2\right) + H_1 - \frac{\gamma_p}{\rho+\delta-\theta\alpha_1} H_2.$$

In order to avoid the trivial moment of investment $\tau^* = 0$, we will further suppose that the initial values of the processes satisfy $\pi_0 < p^* I_0$.

If we know the optimal moment for investment, we can find the expected optimal present value for the investor as well as the relevant expected present tax payments into the budgets at different levels. Let us denote the expected present value for the investor under the optimal behavior (i.e. maximum value of the function in (3.11)) as $\mathcal{N}$, let $\mathcal{T}^f = \mathbf{E}\left(T^f_{\tau^*} e^{-\rho\tau^*}\right)$ be the expected present tax payments into the federal budget under the optimal behavior, and $\mathcal{T}^r = \mathbf{E}\left(T^r_{\tau^*} e^{-\rho\tau^*}\right)$ be the similar value for tax payments into the regional budget, $\mathcal{B} = \mathcal{B}_{\tau^*}$ be the expected present tax burden under the optimal behavior of investor. Furthermore, it is also interesting to study the expected time of investment waiting $\mathbf{E}\tau^*$.

Using Theorem 4 from Chapter 7, we can obtain the following formulas for these indicators.

Theorem 2. *Let the amount of required investments I_t be described by the process (4.1), and values added π_t by the relation (4.3). Assume that $\tilde{\sigma} > 0$, $\alpha_2 - \frac{1}{2}\sigma_2^2 \geq \alpha_1 - \frac{1}{2}\sigma_1^2$, and $\rho > \max(\alpha_1, \alpha_2)$. Then, the following formulas hold:*

1) $\mathcal{N} = I_0 \left(\frac{\pi_0}{I_0 p^*}\right)^\beta \left(1 - H_1 + \frac{\gamma_p}{\rho+\delta-\theta\alpha_1} H_2\right) \cdot \frac{1}{\beta-1}$*;*

2) $\mathcal{T}^f = I_0 \left(\frac{\pi_0}{I_0 p^*}\right)^\beta \left\{ \frac{\gamma_{va}^f + \gamma_i^f(1-\mu) + \tilde{\gamma}_s^f \tilde{\mu}}{\rho - \alpha_2} p^* - \gamma_i^f \left[\frac{\gamma_p}{\rho - \theta\alpha_1} + \left(1 - \frac{\gamma_p}{\rho - \theta\alpha_1}\right) K_0 \right] \right\},$

3) $\mathcal{T}^r = I_0 \left(\frac{\pi_0}{I_0 p^*}\right)^\beta \left\{ \frac{\gamma_{va}^r + (\gamma_i^0 + \Delta\gamma_i e^{-(\rho-\alpha_2)\nu})(1-\mu) + \tilde{\gamma}_s^r \tilde{\mu}}{\rho - \alpha_2} p^* - H_1^0 + \frac{\gamma_p}{\rho - \theta\alpha_1} H_2^0 \right\};$

4) $\mathcal{B} = \gamma_{va} + \gamma_i(1-\mu) + \tilde{\gamma}_s\tilde{\mu} - (\rho - \alpha_2) \left[\frac{\gamma_p}{\rho - \theta\alpha_1}(1 - K_0) - \gamma_i K_0 \right] (p^*)^{-1};$

5) $\mathbf{E}\tau^* = (\alpha_2 - \frac{1}{2}\sigma_2^2 - \alpha_1 + \frac{1}{2}\sigma_1^2)^{-1} \log\left(\frac{I_0 p^*}{\pi_0}\right);$

where p^* *is defined in Theorem 1, and* $K_0 = \psi A_0^0 + (1-\psi) B_0^0$.

(See the proofs of Theorems 1 and 2 in Chapter 7.)

4.4. Comparative statics. The mutual influence of depreciation mechanisms and tax holidays on investment activity

In the previous Section it was obtained an optimal rule of investment, and on this base were derived the formulas for the various economic indices related to given investment project. The main feature of those formulas was the explicit (analytical) description of the dependence of indices of more than twenty different parameters.

The dependence of some parameters is simple (for example, the monotonicity). But the mutual influence of certain parameters can lead to far less obvious analytical results.

For example, taken separately tax holidays and accelerated depreciation are fiscal incentives, stimulate investment activity and especially favor the earlier entry of investor. But, as it will be demonstrated below, the simultaneous application of tax holidays and accelerated depreciation can lead (under certain conditions) to decreasing investment activity. It is related to the fact that, under tax holidays, profit tax is absent and depreciation allowances act in vain[11].

[11] Some developing countries (e.g. Malaysia, Côte d'Ivoire) permits to delay depreciation allowances until end of tax holidays (see Mintz, 1990)

In order to investigate such a phenomena let us study the dependence of optimal investment level p^* which determines the optimal moment for investor entry on tax holidays and depreciation.

In order to simplify the formulas we will ignore the property tax which will be considered as zero (i.e. $\gamma_p = 0$). In this case the optimal level of investment is the following

$$p^* = (1 - H_1)\tilde{p}, \quad \text{where } \tilde{p} = \frac{\rho + \delta - \alpha_2}{(1-\mu)(1-\widehat{\gamma}_i)} \cdot \frac{\beta}{\beta - 1}, \tag{4.11}$$

$\widehat{\gamma}_i = \bar{\gamma}_i + \Delta\gamma_i e^{-(\rho+\delta-\alpha_2)\nu}$, $H_1 = \bar{\gamma}_i[\psi A_0 + (1-\psi)B_0] + \Delta\gamma_i[\psi A_\nu + (1-\psi)B_\nu]$, and A_0, A_ν, B_0, B_ν are defined in (4.5).

Thus we have

$$\begin{aligned}
\frac{\partial p^*}{\partial \nu} &= -\tilde{p}\frac{\partial H_1}{\partial \nu} + \frac{(1-H_1)\tilde{p}}{1-\widehat{\gamma}_i} \cdot \frac{\partial \widehat{\gamma}_i}{\partial \nu} = \tilde{p}\Delta\gamma_i e^{-(\rho+\delta-\theta\alpha_1)\nu}[\psi a_\nu + (1-\psi)b_\nu] \\
&- \frac{(1-H_1)\tilde{p}}{1-\widehat{\gamma}_i}\Delta\gamma_i(\rho+\delta-\alpha_2)e^{-(\rho+\delta-\alpha_2)\nu} \\
&= \tilde{p}\Delta\gamma_i e^{-(\rho+\delta)\nu}\left[c_\nu e^{\theta\alpha_1\nu} - \frac{\beta-1}{\beta}(1-\mu)p^* e^{\alpha_2\nu}\right],
\end{aligned}$$

where $c_\nu = \psi a_\nu + (1-\psi)b_\nu$.

The sign of the derivative $\dfrac{\partial p^*}{\partial \nu}$ is thus defined by the sign of the following expression

$$c_\nu e^{\theta\alpha_1\nu} - \frac{\beta-1}{\beta}(1-\mu)p^* e^{\alpha_2\nu}$$

or after a multiplication by positive value $I\tau^*$, by the sign of expression

$$\Delta_\nu = I_{\tau^*} c_\nu e^{\theta\alpha_1\nu} - \frac{\beta-1}{\beta}(1-\mu)\pi_{\tau^*} e^{\alpha_2\nu}$$

(although this expression is a random variable its sign is not random!).

The last expression have a clear economic interpretation. Let us notice, that according to formulas (3.5) and (4.5), $I_{\tau^*} c_\nu e^{\theta\alpha_1\nu} = \mathbf{E}\left(D^{\tau^*}_{\tau^*+\nu} \mid \mathcal{F}_{\tau^*}\right)$ is the forecasting (at moment τ^*) depreciation deduction at the moment $\tau^* + \nu$, and due to assumption (4.3) $\pi_{\tau^*} e^{\alpha_2\nu} = \mathbf{E}\left(\pi^{\tau^*}_{\tau^*+\nu} \mid \mathcal{F}_{\tau^*}\right)$ is the forecasting (made at moment τ^*) value added at the moment $\tau^* + \nu$.

Thus, the sign of $\dfrac{\partial p^*}{\partial \nu}$ depends on the relation between the predicted depreciation charge and predicted value added at the moment of ending tax holidays. If the value added prevails over depreciation charge, i.e $\Delta_\nu < 0$, then the increase in the duration of tax holidays implies drop in the optimal investment level, and therefore, the time of investment waiting decreases. In other words, the increase of tax exemptions in this case stimulates investment activity. However, if depreciation charge prevails over value added and a contrary inequality holds, then we observe a very surprising result: the increase of the duration of tax holidays implies a later entry of investor (since level p^* increases), i.e. *decrease* in investment activity.

Let us illustrate our theoretical arguments by numerical examples.

Since an optimal moment of investment is a random variable, it is convenient to compare the expected delay in the investment moments for various tax holidays (for a simplification tax holidays are assumed to be full). Let us suppose that τ_ν^* is the optimal moment of investment when tax holidays equal ν and $\Delta\tau = \mathbf{E}\tau_0^* - \mathbf{E}\tau_\nu^*$ is the expected speed up of investment when tax holidays are ν (related to zero tax holidays).

Table 4.1 exposes the results of the calculations of $\Delta\tau$ for different values of tax holidays ν and for various shares of active assets ψ in the investment. Has been considered in this calculation a project of investment with parameters $\alpha_2 = 2\%,\ \sigma_2 = 0.1, \alpha_1 = \sigma_1 = 0$ and discount $\rho = 10\%$ (risk is absent). We chose the non-linear depreciation method with rates 35% (for active part of assets) and 3% (for inactive part).

Table 4.1

ν	$\psi = 0.9$	$\psi = 0.5$
1	-3.2	-1.2
2	-4.6	-1.5
3	-5.1	-1.3
4	-5.1	-0.9
6	-4.2	0.3

The calculations show that when tax holidays increase the delay in the investment can be significant, particularly when the share of active assets is sufficiently high. Let us remark that if we choose 10% for depreciation rate of active assets, then the value $\Delta\tau$ will already be positive for any ν (even for $\psi = 0.9$).

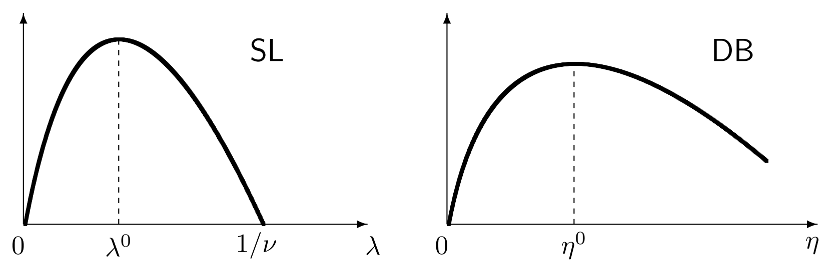

Figure 1. *Dependence of A on the depreciation rate*

Let us study now the dependence of p^* on depreciation rate[12]. From (4.11) we have that this dependence can be presented as a superposition of two functions. The first one is $p^* = p^*(A)$, where $A = \bar{\gamma}_i A_0 + \Delta\gamma_i A_\nu$ is a weighted sum of integrated discounted depreciation charges.

The second function defines the dependence of A on the depreciation rate. It is easy to see that $p^*(A)$ is monotone decreasing function (in A).

As for the second function, for which even in the case of full tax holidays ($\bar{\gamma}i = 0$) and for any authorized depreciation method (linear or non-linear) the dependence of A on the depreciation rate is non-monotone. One can see the graph of A as a function of the depreciation rate for both linear (see (3.6)) and non-linear (see (3.7)) methods on Figure 1.

As one can see on those two graphs, A has a maximum in a depreciation rate: this maximum is attained for the linear method at the point λ^0 which is the root of equation

$$1 + \rho/\lambda = e^{\rho/\lambda - \rho\nu};$$

and for the non-linear method at the point η^0, which is the root of equation

$$\frac{\rho}{\rho + \eta} = \eta\nu.$$

Let us notice that with a decrease in tax holidays the points of maximum λ^0 and η^0 increase and when tax holidays equal zero the intervals of decreasing are absent, i.e λ^0=η^0=∞ when $\nu = 0$.

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[12] We will bound our considerations by a depreciation of active assets only

Thus, under full tax holidays type of dependence of the investment level p^* on depreciation rate significantly depends on the value of depreciation rate itself. If the rate of depreciation does not exceed a certain "critical" level the increase in the depreciation rate leads to a decrease in the level p^* which means that the investor arrives earlier. And, if depreciation rate is sufficiently high (i.e. exceeds the critical level) the increase in depreciation leads to a contrary effect (the increase in the level p^*) which means decreasing in investment activity, i.e later arrival of the investor.

Consequently, the introduction of accelerated depreciation does not always stimulate investor under the presence of tax holidays. Moreover, if the rate of depreciation is high the accelerated depreciation leads to the contrary effect — a later arrival of investor.

The dependence of A on the depreciation rate in the case of non-full tax holidays (when $\bar{\gamma}_i$ is non-zero) becomes far more complicated. In that case A can have several points of local extremum which are the roots for the following equation:

$$\bar{\gamma}_i\rho + \Delta\gamma_i e^{-(\eta+\rho)\nu}[\rho - \eta\nu(\eta+\nu)] = 0.$$

More than two areas of monotonicity appear and the description of the influence of the depreciation rate upon the investment activity becomes a very difficult case even at the qualitative level.

5. OPTIMIZATION APPROACH TO REGIONAL STIMULATION OF INVESTMENT PROJECTS BY MEANS OF DEPRECIATION MECHANISM

In this Chapter we consider the problem of attraction of investment on the projects necessary for the region.

The principal hypothesis that we will take into account consists in the fact that the region, developing the investment project (particularly choosing depreciation policy), believes that investor will behave as described in the model presented above (in Chapter 3). In other words the absence of investment at a given moment of time (investment waiting) is treated as the optimal investor's behavior, who decides to observe the environment and postpones investment in the project. Knowing the dependence of optimal investment rule on the parameters of tax system, and particularly on depreciation policy, the region choose this policy, following its own criteria.

Despite the fact that the region can be interested in numerous aspects to the realization of the project, in this Chapter we will consider that the region only follows a fiscal goals. The optimal depreciation policy will be found that provides maximal expected tax payments into the regional budget and its investigation will be conducted.

5.1. Formulation of the regional stimulation problem

Let us now describe the model of stimulation of the investment project at the regional level.

The base of this model is the hypothesis of rational behavior of the investor (optimal investment timing in the sense of NPV criteria). Supposing that the potential investor behaves rationally the region considers the absence of investment at a given moment of time as the exceeding of optimal threshold p^* over the observed ratio π_t/I_t. The optimal threshold $p^* = p^*(D)$, which is function of the depreciation policy D, defines, due to Theorem 1, the optimal moment of investment $\tau^* = \tau^*(D)$. Thus for each depreciation policy D the region is able to define the expected revenues into the regional budget after the realization of the project (under optimal investor behavior) — see Theorem 2 in Section 4.3:

$$\mathcal{T}^r(D) = I_0 \left(\frac{\pi_0}{I_0 p^*}\right)^\beta \left\{ \frac{\gamma_{va}^r + (\gamma_i^0 + \Delta\gamma_i e^{-(\rho-\alpha_2)\nu})(1-\mu) + \gamma_s^r \tilde{\mu}}{\rho - \alpha_2} p^* \right.$$
$$\left. - H_1^0 + \frac{\gamma_p}{\rho - \theta\alpha_1} H_2^0 \right\}, \tag{5.1}$$

where $H_1^0 = H_1^0(D),\ H_2^0 = H_2^0(D)$.

Let us suppose (in the present Chapter) that the region, having the possibility of choice of the depreciation policy for the investment project, follows the principle of optimality. This means that among the available depreciation policies (which are available for the region) the region chooses the one that maximizes expected discounted tax payments into the regional budget from the created enterprise (5.1). Thus, the problem of stimulation is deducted to the problem of optimization by the region of the functional $\mathcal{T}^r(D)$ on a certain available class of depreciation policies $\mathcal{D}$:

$$\mathcal{T}^r(D) \to \max_{D \in \mathcal{D}}. \tag{5.2}$$

The available classes of depreciation policies $\mathcal{D}$ can be defined by different depreciation methods (linear, non-linear), and the number of additional restrictions. Those restrictions can come, for example, from the competition between regions in order to "attract" the potential investor and are specified by lower bound on NPV of the investor. Another type of restrictions can be linked to the continuous (in average) supply of tax payments of the enterprise into the regional budget for any time after investment, i.e. the gross profit $Z_t^{\tau^*}$ (defined by the formula (3.1)) will be not less (in average) than a given level. Investigation of similar problems (with restrictions on expected gross profits) was started in Arkin and Slastnikov (2000b).

5.2. Optimal depreciation policy

In this Section we find an optimal depreciation policy (depreciation density) that provides maximal expected present tax payments from the project into the regional budget. Since changing in depreciation method and rates is allowed as a rule for active part of assets only, we will identify a depreciation policy with a depreciation density of active part of assets $(a_t,\ t \geq 0)$. The depreciation density of inactive assets $(b_t,\ t \geq 0)$ will be considered as fixed.

Let $\mathcal{D}$ be a given set of all available (for the regional control) depreciation policies D. For any depreciation policy $D = (a_t,\ t \geq 0)$ one can define the integral from discounted depreciation density

$$A = A(D) = \int_0^\infty a_t e^{-(\rho-\theta\alpha_1)t}\, dt,$$

similar as in previous Sections.

Let us denote

$$\min_{D\in\mathcal{D}} A(D) = \underline{A}, \quad \max_{D\in\mathcal{D}} A(D) = \bar{A}. \text{ [13]}$$

We assume that the set of available depreciation policies is enough "rich" in the following sense.

(C) For any value $a,\ \underline{A} < a < \bar{A}$ there exists depreciation policy $D \in \mathcal{D}$ such that $A(D) = a$. [14]

In other words, set $\{a:\ a = A(D),\ D \in \mathcal{D}\}$ will be the interval $[\underline{A}, \bar{A}]$.

In order to obtain explicit formulas for optimal depreciation policy we will assume that both tax holidays and risk process are absent, i.e. $\nu = 0$ and

[13] If *min* and/or *max* are not attained for some sets $\mathcal{D}$, one can say about *inf* and/or *sup*

[14] It is valid, e.g., if set $\mathcal{D}$ consists of uniform or exponential densities

$\delta = 0$. In this case the expression for present tax payments (5.1) can be written as follows:

$$\mathcal{T}^r(D) = I_0 \left(\frac{\pi_0}{I_0 \tilde{p}}\right)^{\beta} (1-u)^{-\beta} [q^r(1-u) - h_1 u + h_2], \tag{5.3}$$

where $u = u(D) = (\gamma_i + \Gamma)K - \Gamma$, $\Gamma = \dfrac{\gamma_p(1-\gamma_i)}{\rho - \theta\alpha_1}$, $K = K(D) = \psi A(D) + (1-\psi)B$,

$q^r = \dfrac{\gamma_{va}^r + \gamma_i^r(1-\mu) + \tilde{\gamma}_s^r \tilde{\mu}}{(1-\mu)(1-\gamma_i)} \cdot \dfrac{\beta}{\beta - 1}$, $\quad B = \displaystyle\int_0^\infty b_t e^{-(\rho - \theta\alpha_1)t}\, dt$,

$h_1 = \dfrac{\gamma_i^r + \Gamma^r}{\gamma_i + \Gamma}$, $\quad h_2 = \dfrac{\gamma_i \Gamma^r - \gamma_i^r \Gamma}{\gamma_i + \Gamma}$, $\quad \Gamma^r = \dfrac{\gamma_p(1-\gamma_i^r)}{\rho - \theta\alpha_1}$.

Thus the problem of finding an optimal depreciation policy by the region (5.2) is reduced (due to formula (5.3)) to a more simple problem of maximization of a function at some interval, namely,

$$g(u) \to \max_{\underline{u} \le u \le \bar{u}}, \tag{5.4}$$

where $g(u) = (1-u)^{-\beta}[q^r(1-u) - h_1 u + h_2]$,
$\underline{u} = (\gamma_i + \Gamma)[\psi \underline{A} + (1-\psi)B] - \Gamma$, $\quad \bar{u} = (\gamma_i + \Gamma)[\psi \bar{A} + (1-\psi)B] - \Gamma$.

Differentiating $g(u)$ we have:

$$\begin{aligned} g'(u) &= \beta(1-u)^{-\beta-1}[q^r(1-u) - h_1 u + h_2] - (1-u)^{-\beta}(q^r + h_1) \\ &= (1-u)^{-\beta-1} G(u), \qquad (5.5) \\ \text{where } G(u) &= \beta\,[q^r(1-u) - h_1 u + h_2] - (1-u)\,(q^r + h_1) \\ &= (\beta - 1) q^r (1-u) - \beta h_1 u - (1-u) h_1 + \beta h_2 \\ &= (\beta Q - h_1)(1-u) - \beta h_1 u + \beta h_2, \qquad (5.6) \end{aligned}$$

and $Q = q^r \dfrac{\beta - 1}{\beta} = \dfrac{\gamma_{va}^r + \gamma_i^r(1-\mu) + \tilde{\gamma}_s^r \tilde{\mu}}{(1-\mu)(1-\gamma_i)}$.

$G(u)$ is decreasing function since $G'(u) = -\beta Q - (\beta - 1)h_1 < 0$.

Let us pass to solving the maximization problem (5.4). Let u^* denote an optimal solution in (5.4).

If $G(\underline{u}) \le 0$ then $G(u) \le 0$ whenever $u > \underline{u}$, and due to (5.5) $g(u)$ decreases in u, therefore $u^* = \underline{u}$, hence optimal depreciation policy D^* satisfies $A(D^*) = \underline{A}$. Formula (5.6) implies that condition $G(\underline{u}) \le 0$ is equivalent to

$$\beta[Q - (Q + h_1)\underline{u} + h_2] \le (1 - \underline{u}) h_1. \tag{5.7}$$

Similarly one can obtain that $G(u) > 0$ whenever $u \le \bar{u}$ is equivalent to

$$\beta[Q - (Q + h_1)\bar{u} + h_2] \ge (1 - \bar{u})h_1. \tag{5.8}$$

Hence, in this case $g(u)$ increases in u, and therefore, $u^* = \bar{u}$.

If both inequalities (5.7) and (5.8) are not held, then $g(u)$ attains maximum at the point u^* such that $G(u^*) = 0$, and

$$u^* = \frac{\beta Q - h_1 + \beta h_2}{\beta Q + \beta h_1 - h_1}. \tag{5.9}$$

Collect the above results we have the following theorem.

Theorem 3. *Let condition* (**C**) *holds. Then depreciation policy* $D^* = (a_t^*,\ t \ge 0)$ *is optimal if and only if the discounted depreciation density* $A^* = \int\limits_0^\infty a_t^* e^{-(\rho - \theta\alpha_1)t}\, dt$ *satisfies the following relations:*

$$A^* = \begin{cases} \underline{A}, & \textit{if (5.7) holds} \\ \bar{A}, & \textit{if (5.8) holds} \\ \left[\dfrac{u^* + \Gamma}{\gamma_i + \Gamma} - (1 - \psi)B\right]/\psi, & \textit{else} \end{cases}$$

where u^* *is defined in (5.9).*

Now we give the relevant results for the major classes of depreciation methods, specified in Section 3.2.

For the linear method with depreciation rate λ (see (3.6)) we have

$$A = A^{SL}(\lambda) = \frac{\lambda}{\rho - \theta\alpha_1}\left(1 - e^{-(\rho - \theta\alpha_1)/\lambda}\right).$$

Note that $A^{SL}(\lambda)$ is increasing (in λ) function.

Assume that available depreciation rates have to place between two boundaries $\underline{\lambda}$ and $\bar{\lambda}$, i.e. $0 < \underline{\lambda} \le \lambda \le \bar{\lambda} < \infty$. As one can easy see the condition (**C**) is valid for this case. Let us denote

$$\underline{A} = A^{SL}(\underline{\lambda}), \quad \bar{A} = A^{SL}(\bar{\lambda}).$$

Corollary 1. *An optimal depreciation rate λ for SL method under restrictions $\underline{\lambda} \le \lambda \le \bar{\lambda}$ has the following form*

$$\lambda^* = \begin{cases} \underline{\lambda}, & \textit{if (5.7) holds} \\ \bar{\lambda}, & \textit{if (5.8) holds} \\ \tilde{\lambda}, & \textit{else} \end{cases}$$

where $\tilde{\lambda}$ is a root of the equation $\psi A^{SL}(\lambda) = \dfrac{u^* + \Gamma}{\gamma_i + \Gamma} - (1-\psi)B$, and u^* is defined in (5.9).

For the non-linear method with rate η (see (3.7)) we have

$$A = A^{DB}(\eta) = \frac{\eta}{\rho - \theta\alpha_1 + \eta}.$$

The function $A^{DB}(\eta)$ also increases (in η).

If there are certain restrictions on depreciation rate $\underline{\eta} \le \eta \le \bar{\eta}$, then (**C**) is also satisfied. Similar to previous considerations let us denote

$$\underline{A} = A^{DB}(\underline{\eta}), \quad \bar{A} = A^{DB}(\bar{\eta}).$$

Corollary 2. *An optimal depreciation rate η for DB method under restrictions $\underline{\eta} \le \eta \le \bar{\eta}$ is the following*

$$\eta^* = \begin{cases} \underline{\eta}, & \textit{if (5.7) holds} \\ \bar{\eta}, & \textit{if (5.8) holds} \\ (\rho - \theta\alpha_1)\tilde{A}/(\psi - \tilde{A}), & \textit{else} \end{cases}$$

where $\tilde{A} = \dfrac{u^* + \Gamma}{\gamma_i + \Gamma} - (1-\psi)B$, and u^* is defined in (5.9).

5.3. Taking into account restrictions on depreciation policy

In previous Section where we derive an optimal depreciation policy we took into account some exogenous restrictions at the depreciation rate. However, there are restrictions which issue from the model.

One of these will be studied below and concerns gross profits (tax base for corporate income tax) Z_t^τ.

It is well known that the presence of negative gross profits for a long period of time is undesirable both for shareholders and for tax authorities. Lack of supplies in budget may have a negative influence on investment climate.

In order to avoid this situation we will consider the following condition of *a positivity (non-negativity) of the gross profit in average* (PGPA) at any moment after investment under optimal investor behavior:

$$\mathbf{E}(Z_t^{\tau^*} \mid \mathcal{F}_{\tau^*}) \geq 0, \text{ (a.s.)} \quad \text{for all} \quad t \geq \tau^* \tag{5.10}$$

Remembering specification of Z_t^τ from (3.1) and assumptions (4.3)–(4.4), we have that condition (5.10) is equivalent to the following: for all $t \geq 0$

$$(1-\mu)\pi_{\tau^*} e^{\alpha_2 t} \geq I_{\tau^*} e^{\theta\alpha_1 t} \left\{ \psi a_t + (1-\psi) b_t + \gamma_p [1 - \psi \widehat{a}_t - (1-\psi)\widehat{b}_t] \right\} \quad \text{(a.s.)},$$

where $\widehat{a}_t$, $\widehat{b}_t$ are defined in (3.9).

Applying Theorem 1 one can reduce (5.10) to the inequalities: for all $t \geq 0$

$$\begin{aligned}(1-u)\frac{\beta}{\beta-1} \cdot \frac{\rho-\alpha_2}{1-\gamma_i} e^{(\alpha_2-\theta\alpha_1)t} \ &\geq \ \psi a_t + (1-\psi) b_t \\ &+ \ \gamma_p [1 - \psi \widehat{a}_t - (1-\psi)\widehat{b}_t], \end{aligned} \tag{5.11}$$

where u is defined in (5.3).

Inequalities (5.11) induce certain restrictions at depreciation policy. We study such restrictions both for linear and non-linear methods.

Non-linear method (3.7)

Let η_a be a depreciation rate of active assets, η_b be a depreciation rate of inactive assets. Then

$$a_t = \eta_a e^{-\eta_a t}, \ b_t = \eta_b e^{-\eta_b t}, \ K = \psi \frac{\eta_a}{\eta_a + \rho - \theta\alpha_1} + (1-\psi)\frac{\eta_b}{\eta_b + \rho - \theta\alpha_1},$$

and (5.11) can be rewritten as follows: for all $t \geq 0$

$$(1-u)\frac{\beta}{\beta-1} \cdot \frac{\rho-\alpha_2}{1-\gamma_i} e^{(\alpha_2-\theta\alpha_1)t} \geq \psi(\gamma_p + \eta_a) e^{-\eta_a t} + (1-\psi)(\gamma_p + \eta_b) e^{-\eta_b t}. \tag{5.12}$$

If we suppose that $\eta_a \geq \eta_b$ (depreciation rate for active assets is greater than depreciation rate for active assets), then under the additional (nonsignificant) condition $\alpha_2 + \eta_b \geq \theta\alpha_1$ inequality (5.12) holds for all $t \geq 0$ if and only if

$$(1-u)\frac{\beta}{\beta-1} \cdot \frac{\rho-\alpha_2}{1-\gamma_i} \geq \gamma_p + \psi(\eta_a - \eta_b). \tag{5.13}$$

It is easy to see that inequality (5.13) will be valid for $\eta \le \eta_0$ where η_0 is a root of the quadratic equation

$$(1-u)\frac{\beta}{\beta-1}\cdot\frac{\rho-\alpha_2}{1-\gamma_i}=\gamma_p+\psi(\eta_a-\eta_b). \tag{5.14}$$

Thus under the PGPA condition (5.10) the optimal depreciation rate for DB method equals to $\eta^{**}=\min(\eta^*,\eta_0)$ where η^* is defined in Corollary 2, and η_0 is a root of the equation (5.14).

Linear method (3.6)

For this case let λ_a be a depreciation rate of active assets, λ_b be a depreciation rate of inactive assets. Then $\widehat{a}_t=\min(\lambda_a t,1),\ \widehat{b}_t=\min(\lambda_b t,1)$,

$$K=\psi\frac{\lambda_a}{\rho-\theta\alpha_1}\left(1-e^{-(\rho-\theta\alpha_1)/\lambda_a}\right)+(1-\psi)\frac{\lambda_b}{\rho-\theta\alpha_1}\left(1-e^{-(\rho-\theta\alpha_1)/\lambda_b}\right). \tag{5.15}$$

Suppose that $\alpha_2\ge\theta\alpha_1$. Then left side of inequality (5.11) increases in t, and right side decreases in t. Therefore, (5.11) holds for all $t\ge 0$ if and only if it holds for $t=0$, i.e.

$$(1-u)\frac{\beta}{\beta-1}\cdot\frac{\rho-\alpha_2}{1-\gamma_i}\ge\gamma_p+\psi\lambda_a+(1-\psi)\lambda_b. \tag{5.16}$$

As in linear case the inequality (5.16) is valid for $\lambda_a\le\lambda_0$, where λ_0 is a root of the equation

$$(1-u)\frac{\beta}{\beta-1}\cdot\frac{\rho-\alpha_2}{1-\gamma_i}=\gamma_p+\psi\lambda_a+(1-\psi)\lambda_b. \tag{5.17}$$

Thus, under PGPA condition (5.10) an optimal depreciation rate for linear method is $\lambda^{**}=\min(\lambda^*,\lambda_0)$, where λ^* is specified in Corollary 1 from Theorem 3, and λ_0 is a root of the equation (5.17).

Let us note that instead of positivity of gross profit in average one can consider more general condition: expected gross profit at any moment after investment has to be no less than a given value. In this way it is not difficult to obtain relevant modifications of the boundaries $\lambda_0,\ \eta_0$ for the corresponding restrictions in both linear and non-linear methods.

5.4. Some numeric examples

In this Section we will present some results of the calculation of optimal depreciation policy on real adjusted data.

In order to be clear, let the amount of necessary investment be constant (i.e. $\alpha_1 = \sigma_1 = 0$). As the "reasonable" area of parameters of the model we will take $\alpha_2 \sim -1\% - 2\%,\ \sigma_2 \sim 0.1 - 0.15$, the discount rate will be $\rho \sim 10\% - 20\%$ (all parameters are annual).

In tables below, we will present the values of optimal depreciation rate for linear and non-linear methods for different values of parameters of investment project (expected rate of change in value added, volatility and share of active assets) and discount. In order to simplify the situation, we will assume trivial the exogenous restrictions on the depreciation rate, i.e. $\underline{\lambda} = \underline{\eta} = 0,\ \bar{\lambda} = \bar{\eta} = \infty$. In every tables we will denote λ^* and η^* as the optimal rate of investment for SL and DB methods (Corollaries 1 and 2), λ_0 and η_0 – as the corresponding upper bounds obtained from the conditions of positivity of gross profit in average (5.10), i.e. the roots of equations (5.17) and (5.14).

Let us remark that conformably to recently adopted laws is integrated to the regional budget a part of the corporate profit tax (at the rate of 16.5%[15], property tax (at the rate of 2%) and personal income tax (13% from payroll).

In Table 5.1 we show the dependence of the optimal depreciation rate for linear and non-linear methods λ^* and η^*, and the corresponding upper bounds λ_0 and η_0 for the condition of positivity of gross profit (5.10) on the average rate of change of the value added of the project α_2. Volatility of the project equals $\sigma_2 = 0.12$, discount equals 10%, the share of active assets in the initial investment is $\psi = 0.9$, depreciation rate of inactive part of assets equals 5% (for non-linear method), or nearly 3.5% (for linear method), and the share of payroll (relatively to value added) equals 30%.

Table 5.1

α_2	λ^*	λ_0	η^*	η_0
-1%	0.47	**0.15**	0.90	**0.21**
0%	0.24	**0.14**	0.45	**0.20**
1%	**0.14**	0.14	0.25	**0.19**
2%	**0.08**	0.13	**0.13**	0.19

[15] Taking into account the part of the tax dedicated to the local budgets

Let us emphasize that in the present and following tables, boldface values stands for the minimum of optimal depreciation rate and the corresponding upper bounds of restrictions (5.10), i.e. for the optimal depreciation rate under the condition of positivity of gross profit in average.

As we can see in this table, the optimal depreciation rate for a project with an average rate of change in value added α_2 has to be relatively high and exceeds the restrictions consecutive to condition (5.10). Therefore, for those projects the condition of positivity of the gross profit plays a primary role for the choice of best depreciation. When the rate of change of value added increases, the optimal rate of depreciation decreases and begins to satisfy the condition (5.10).

The following table shows the dependence of the same indicators on the share of active assets ψ. It is taken a project with parameters $\alpha_2 = 1\%,\ \sigma_2 = 0.12$, other parameters are the same as above.

Table 5.2

ψ	λ^*	λ_0	η^*	η_0
0.9	**0.14**	0.14	0.25	**0.19**
0.8	0.17	**0.15**	0.32	**0.21**
0.7	0.24	**0.17**	0.46	**0.24**
0.6	0.47	**0.20**	0.91	**0.27**

The increase of the optimal depreciation rate under decrease of ψ is not difficult to obtain directly from Corollaries 1 and 2 of Theorem 3.

Table 5.3 demonstrates the dependence of above mentioned indicators on the discount ρ and the average rate of change in value added α_2. In this case the volatility is $\sigma_2 = 0.15$, the share of active assets is $\psi = 0.9$.

Let us notice that the sensitivity of optimal depreciation rate from the volatility can be high enough, and under small changes of volatility the optimal depreciation rate can move on the frontier of those restriction.

But in many cases as it has been shown by our calculations, the value of optimal depreciation rate is quite reasonable and corresponds quite well to the practically used depreciation rates.

Table 5.3

ρ	λ^*	λ_0	η^*	η_0
$(\alpha_2 = -1\%)$				
10%	0.21	**0.16**	0.39	**0.22**
15%	0.49	**0.23**	0.91	**0.29**
20%	1.08	**0.30**	2.01	**0.36**
$(\alpha_2 = 0\%)$				
10%	**0.14**	0.15	0.24	**0.21**
15%	0.32	**0.22**	0.58	**0.28**
20%	0.65	**0.29**	1.21	**0.36**
$(\alpha_2 = 1\%)$				
10%	**0.09**	0.15	**0.15**	0.20
15%	**0.21**	0.22	0.38	**0.28**
20%	0.44	**0.29**	0.80	**0.35**

5.5. What does optimal depreciation policy bring to the federal budget and investor?

The calculations presented in the previous section do not give however any idea about the degree to which the optimal depreciation rate can influence discounted tax payments into the federal budget and other indicators linked to a given investment project (indicators which were discussed in Section 4.4).

In this Section we show how the optimal depreciation rate influences tax revenue into the federal budget and also the net present value of the investor (NPV). As a *relative estimation of efficiency* we will consider the ratio of the indicator corresponding to the optimal depreciation rate to an indicator corresponding to a "reference" depreciation rate. Since indicators depend on the depreciation policy only by the integral of the discounted depreciation density, in order to be clear we will only study the linear method of depreciation in this Section. In this case, as a "reference" depreciation rate of active assets, we will take $\lambda^0 = 20\%$, which corresponds to the current practice.

Remember that starting from Section 5.3 we suppose the absence of both tax holidays and risk process. In that case the formulas of Theorem 2 for expected present tax payments from created enterprise into the budgets and the expected NPV of investor can be simplified. The expression for presented taxes into the regional budget $\mathcal{T}^r(\lambda)$[16] is given in (5.3). For tax revenues into the federal budget $\mathcal{T}^f(\lambda)$ and the NPV of investor $\mathcal{N}(\lambda)$ are obtained the following formulas:

$$\mathcal{T}^f(\lambda) = I_0 \left(\frac{\pi_0}{I_0 \tilde{p}}\right)^{\beta} (1-u)^{-\beta}[q^f(1-u) - h_3 u - h_4], \tag{5.18}$$

$$\mathcal{N}(\lambda) = \frac{I_0}{\beta - 1} \left(\frac{\pi_0}{I_0 \tilde{p}}\right)^{\beta} (1-u)^{-\beta+1}, \tag{5.19}$$

where $q^f = \dfrac{\gamma_{va}^f + \gamma_i^f(1-\mu) + \tilde{\gamma}_s^f \tilde{\mu}}{(1-\mu)(1-\gamma_i)} \cdot \dfrac{\beta}{\beta-1}$, $\quad u = u(\lambda)$ is specified in (5.3),
$h_3 = \dfrac{\gamma_i^f}{\gamma_i + \Gamma}\left(1 - \dfrac{\gamma_p}{\rho - \theta\alpha_1}\right), \quad h_4 = \dfrac{\gamma_i^f}{\gamma_i + \Gamma}\left(\Gamma + \gamma_i \dfrac{\gamma_p}{\rho - \theta\alpha_1}\right).$
If λ^* denotes the optimal depreciation rate, then for the comparative efficiency estimates we have the following expressions:

$$\mathcal{E}_*^r = \frac{\mathcal{T}^r(\lambda^*)}{\mathcal{T}^r(\lambda^0)} = \left(\frac{1-u^0}{1-u^*}\right)^{\beta} \frac{q^r(1-u^*) - h_1 u^* + h_2}{q^r(1-u^0) - h_1 u^0 + h_2}, \tag{5.20}$$

$$\mathcal{E}_*^f = \frac{\mathcal{T}^f(\lambda^*)}{\mathcal{T}^f(\lambda^0)} = \left(\frac{1-u^0}{1-u^*}\right)^{\beta} \frac{q^f(1-u^*) - h_3 u^* - h_4}{q^f(1-u^0) - h_3 u^0 - h_4}, \tag{5.21}$$

$$\mathcal{E}_*^N = \frac{\mathcal{N}(\lambda^*)}{\mathcal{N}(\lambda^0)} = \left(\frac{1-u^0}{1-u^*}\right)^{\beta-1}, \tag{5.22}$$

where u^* corresponds to depreciation rate λ^*, and u^0 – to the rate λ^0 .

We will also estimate the comparative efficiency of a doubled “reference” depreciation $2\lambda^0$ because this accelerated depreciation was adopted in the former Russian tax system. Corresponding indicators are

$$\mathcal{E}_2^r = \frac{\mathcal{T}^r(2\lambda^0)}{\mathcal{T}^r(\lambda^0)} = \left(\frac{1-u^0}{1-u^2}\right)^{\beta} \frac{q^r(1-u^2) - h_1 u^2 + h_2}{q^r(1-u^0) - h_1 u^0 + h_2}, \tag{5.23}$$

[16] In this Section we will pay attention to the dependence of indicators on depreciation rate

$$\mathcal{E}_2^f = \frac{\mathcal{T}^f(2\lambda^0)}{\mathcal{T}^f(\lambda^0)} = \left(\frac{1-u^0}{1-u^2}\right)^{\beta} \frac{q^f(1-u^2) - h_3 u^2 - h_4}{q^f(1-u^0) - h_3 u^0 - h_4}, \tag{5.24}$$

$$\mathcal{E}_2^N = \frac{\mathcal{N}(2\lambda^0)}{\mathcal{N}(\lambda^0)} = \left(\frac{1-u^0}{1-u^2}\right)^{\beta-1}, \tag{5.25}$$

where u^2 corresponds to the "doubled" depreciation rate $2\lambda^0$.

Before presenting the results let us make certain remarks.

The federal budget receives: the part of corporate income tax (at the rate of 7.5%), VAT (20%) and unified social tax (35% of payroll). As previously mentioned other taxes, i.e. a part of corporate income tax, the property tax and the individual income tax are dedicated to the regional budget. This sharing between the different budget levels is presented in the Budget Law 2001.

In order to derive the optimal rate of linear depreciation we use Corollary 1 of Theorem 3 without exogenous restrictions on depreciation rate (i.e. $\underline{\lambda} = 0, \quad \bar{\lambda} = \infty$). The absence of such restrictions can sometime lead either to very low or very high depreciation rates. Economically it is not justifiable as it means a total absence of depreciation or "instantaneous" depreciation (total deduction) of assets (remember that the problem in this Section is treated from a regional point of view). At theoretical level, this "singular" character of optimal depreciation only means that the dependence of present tax payment into the regional budget on depreciation rate is monotone (decreasing if $\lambda^* = 0$, and increasing if $\lambda^* = \infty$). Therefore, this means that in presence of some restrictions on the depreciation rate the region has to choose (in the framework of an optimization approach) such a restriction as an optimal depreciation rate.

The area of the considered parameters is previously discussed. Let us simply mention that the discount in the calculation will be $\rho = 10\%$, and linear depreciation rate of inactive assets is fixed to 3%.

Table 5.4 presents the values of indicators (5.20)-(5.25) for projects with large share of active assets ($\psi = 0.9$), large expected rate of change in value added ($\alpha_2 = 2\%$) and various volatilties ($\alpha_2 = 2\%$) (low and high) with the dependence on the "labor intensity" of the project (wage per unit of value added $\tilde{\mu}$).

In the Table 5.5 similar results are presented for investment project with small rate of change in value added ($\alpha_2 = 0$).

Table 5.4

$\tilde{\mu}$	$\mathcal{E}_*^r$	$\mathcal{E}_2^r$	$\mathcal{E}_*^f$	$\mathcal{E}_2^f$	$\mathcal{E}_*^N$	$\mathcal{E}_2^N$
$(\sigma_2 = 0.05)$						
0.2	1.01	0.97	0.91	1.12	0.90	1.14
0.35	1.01	1.01	1.12	1.13	1.13	1.14
0.5	1.11	1.06	1.32	1.13	1.34	1.14
0.65	1.25	1.11	1.33	1.14	1.34	1.14
$(\sigma_2 = 0.25)$						
0.2	1.15	0.95	0.80	1.03	0.76	1.04
0.35	1.06	0.97	0.82	1.04	0.80	1.04
0.5	1.00	0.99	0.96	1.04	0.96	1.04
0.65	1.05	1.02	1.09	1.04	1.09	1.04

Table 5.5

$\tilde{\mu}$	$\mathcal{E}_*^r$	$\mathcal{E}_2^r$	$\mathcal{E}_*^f$	$\mathcal{E}_2^f$	$\mathcal{E}_*^N$	$\mathcal{E}_2^N$
$(\sigma_2 = 0.15)$						
0.2	1.03	0.95	0.85	1.10	0.83	1.11
0.35	1.00	.99	1.00	1.10	1.00	1.11
0.5	1.06	1.03	1.25	1.10	1.26	1.10
0.65	1.18	1.08	1.25	1.11	1.26	1.11
$(\sigma_2 = 0.25)$						
0.2	1.11	0.94	0.78	1.05	0.74	1.06
0.35	1.04	0.97	0.85	1.05	0.83	1.06
0.5	1.00	1.00	1.04	1.05	1.04	1.06
0.65	1.08	1.03	1.13	1.06	1.13	1.06

Let us notice that there is no sense in analyzing low volatility cases when rates of change of α_2 are small because the probability of investing in the project (i.e. the passing by the process π_t/I_t the optimal level p^*) is very low. For this reason in Table 5.5 are presented the projects with moderate volatility ($\sigma_2 = 0.15$) but not with low one.

Now present the values of the comparative efficiency indicators for a project with moderate share of active assets ($\psi = 0.5$). Unlike the preceding case, we focus our attention on both high and low rates of value added change, while volatility is quite high ($\sigma_2 = 0.25$).

Table 5.6

$\tilde{\mu}$	$\mathcal{E}^r_*$	$\mathcal{E}^r_2$	$\mathcal{E}^f_*$	$\mathcal{E}^f_2$	$\mathcal{E}^N_*$	$\mathcal{E}^N_2$
$(\alpha_2 = 2\%)$						
0.2	1.06	0.99	0.88	1.02	0.86	1.02
0.35	1.02	0.99	0.89	1.02	0.87	1.02
0.5	1.00	1.00	1.04	1.02	1.05	1.02
0.65	1.03	1.01	1.04	1.02	1.05	1.02
$(\alpha_2 = 0)$						
0.2	1.04	0.99	0.86	1.02	0.83	1.03
0.35	1.00	1.00	0.94	1.03	0.93	1.03
0.5	1.01	1.01	1.06	1.03	1.06	1.03
0.65	1.04	1.02	1.06	1.03	1.06	1.03

Conducted calculations, numerous of which are not presented in the paper, allows us to make the following conclusions about the efficiency of the optimal depreciation policy.

First of all, for every "poorly technically rigged" investment projects (with small share of active assets $\psi \sim 0.2$), the optimal rate of depreciation is very high. That means that the maximal admitted rate will be optimal for the region. However for those projects the depreciation of active assets does not play a great role, and optimal depreciation cannot bring additional revenue of more than 1% – 2% to regional budget. The efficiency of optimal depreciation for federal budget and investor NPV is also low.

Technically rigged projects (with share of active assets ψ representing at least half of the initial investment) present a more complicated picture. When the value of "labor intensity" $\tilde{\mu}$ (wage by unit of value added) is low then the optimal depreciation rate is less than "reference" rate. The additional increase of regional budget revenue can attain as far as 10 to 15% (when volatility is high) but rapidly falls with a decrease in volatility. The federal budget and investor receive in that case considerably less revenue (less than 10% – 25%). In parallel, the "double" depreciation policy is efficient and can bring to federal budget and investor an increase of revenue of 10% – 15%. When the share $\tilde{\mu}$ increases (up to about 0.3 – 0.4), then happens an "equalization" of optimal and "reference" depreciation rates. In that case double depreciation is profitable both for federal budget and investor, and can increase their revenue of about 10%. Under further increasing $\tilde{\mu}$ (beyond 0.5) the optimal depreciation rate exceeds the "reference" rate and gives additional revenue both for the federal budget and the investor. In that case the relative efficiency of the optimal depreciation both for the federal budget and investor is higher than for the regional budget. The additional gain both for the federal budget and the investor can attain 30% – 40% (when volatility is low) but significantly decreases with an increase of volatility.

For projects rigged by active assets moderately, the picture is quite similar to the preceding case. Let us simply remark that in that case the influence of depreciation on the relative efficiency becomes a little less important.

Thus, optimal depreciation is able to bring a considerable positive effect for the federal budget and the investor for the projects with the following features:

1) high share of active assets;

2) moderate "labor intensity" (in term of wage per unit of value added);

3) low or moderate volatility.

For such projects, optimal depreciation rate owns an incentive effect as it forces the investor to launch the investment earlier. In other cases, the incentive effect of optimal depreciation rate (for federal budget and investor) is sometimes not present at all.

6. MODEL ANALYSIS OF THE RUSSIAN TAX SYSTEM

6.1. A comparative analysis of the former and the new corporate profit taxation systems

In this Chapter we will use the model presented above in order to lead a comparison between two taxation systems of firms profit in Russia. The first system that will be referred to as the "former" system was in practice in Russia up to December 31, 2001. Its main characteristics relatively to our study were:

1) tax profit rate 35% (11% are included into the federal budget, 19% into regional budgets and 5% into local budgets);
2) tax holidays for new created enterprises for the length of the payback period but not exceeding three years;
3) accelerated depreciation for the active part of assets (at the linear method) with the acceleration rate not exceeding 2 (a higher rate could be applied with the approval of regional fiscal authorities).

From January 1, 2002, has been put into practice the new taxation system of firm profits, which cancelled both tax holidays for newly created enterprises and accelerated depreciation allowances. But profit tax rate decreased to 24% (where 7.5% are dedicated to federal budget, 14.5% to regional budget and 2% to local budgets).

It will be conducted a comparison of the former and the new tax system using the following indicators.

1. Optimal investment threshold p^*, which characterizes the moment of implantation of the investor.
2. Expected optimal tax payments into consolidated budget $\mathcal{T}^*$, discounted at the moment of time equal to 0 (base).
3. NPV of the investor $\mathcal{N}^*$, discounted at the moment of time equal to 0 (base).
4. Expected discounted tax burden $\mathcal{B}^*$ of a created enterprise.

For the calculation purpose we fixed the following values for tax rates:

- VAT $\gamma_{va} = 20\%$;
- property tax $\gamma_p = 2\%$;
- payroll tax $\gamma_s = 35\%$;
- individual income tax $\gamma_{pi} = 13\%$;
- corporate profit tax $\gamma_i = 35\%$ (in the former system) and $\gamma_i = 24\%$ (in the new system)[17].

Parameters of the investment projects are analyzed in intervals $\alpha_1{=}0,\ \sigma_1{=}0$ (constant investment), $\alpha_2 \sim -1\%$ – $2\%,\ \sigma_2 \sim 0.05$ – 0.5, and a discount rate equal to $\rho = 10\%$.

In order to simplify calculation we will postulate that there is no risk process, i.e $\delta = 0$. Depreciation rates have been fixed to 3% for the inactive part of the assets and 20% for the active part of the assets (according to linear method). For the former fiscal system we applied a tax holidays $\nu = 3$ and an acceleration rate equal to $k_a = 2$.

In the tables below values of the ratios of above presented parameters are shown for the former and new systems:

$$R_p = \frac{p^*_{\text{new}}}{p^*_{\text{old}}}, \quad R_T = \frac{\mathcal{T}^*_{\text{new}}}{\mathcal{T}^*_{\text{old}}}, \quad R_N = \frac{\mathcal{N}^*_{\text{new}}}{\mathcal{N}^*_{\text{old}}} \quad R_B = \frac{\mathcal{B}^*_{\text{new}}}{\mathcal{B}^*_{\text{old}}}$$

for different investment projects scenarios.

Investment projects are separated into groups depending on the relative shares of the active part of assets ψ and of payroll in the value added μ. The value ψ can characterize "the technical performance" of the project (a technically performing project is properly rigged with vehicles, mechanical equipment) and the value μ characterizes the "intensity" of labor force within the project (per unit of value added).

In Table 6.1 values of above mentioned ratios are exposed for a properly technically rigged projects with a low payroll ($\psi = 0.9,\ \mu = 0.2$).

In Table 6.2 analogous results are presented for an other extreme case, a poorly technically rigged investment projects with high payroll ($\psi = 0.2,\ \mu = 0.7$).

The results corresponding to the "intermediate case" with moderate values of ψ are not presented.

[17] All rates and others values in this Chapter are presented on an annual basis

Table 6.1

α_2		$\sigma_2 = 0.25$	$\sigma_2 = 0.5$
-1%	R_p	0.85	0.85
	R_T	1.12	1.01
	R_N	1.30	1.11
	R_B	0.86	0.91
0%	R_p	0.84	0.84
	R_T	1.09	1.00
	R_N	1.29	1.12
	R_B	0.86	0.90
1%	R_p	0.83	0.83
	R_T	1.07	0.98
	R_N	1.28	1.13
	R_B	0.85	0.90
2%	R_p	0.83	0.83
	R_T	1.04	0.97
	R_N	1.26	1.13
	R_B	0.85	0.89

Those numerous calculations allow us to present a number of conclusions concerning the comparison of former and new tax systems. Let us emphasize upon the fact that our conclusions only concern investment projects for newly created enterprises for which the specific feature is a possibility to choose the moment of investment.

Properly technically rigged investment projects ($\psi \sim 0.9$). With the new system, investor comes earlier (the optimal investment threshold under the new system is less by 15% – 20% compared to the former system). The difference between fiscal revenues in the budget in the former and new systems decreases with the increase of uncertainty (volatility of the project). With a

Table 6.2

α_2		$\sigma_2 = 0.25$	$\sigma_2 = 0.5$
-1%	R_p	0.96	0.96
	R_T	1.07	1.02
	R_N	1.09	1.04
	R_B	1.00	1.00
0%	R_p	0.95	0.95
	R_T	1.07	1.03
	R_N	1.11	1.06
	R_B	1.00	1.00
1%	R_p	0.94	0.94
	R_T	1.08	1.03
	R_N	1.12	1.07
	R_B	0.99	1.00
2%	R_p	0.93	0.93
	R_T	1.07	1.03
	R_N	1.13	1.08
	R_B	0.99	0.99

moderate volatility (about 0.25) the consolidated budget will receive about 10% to 20% extra revenue under the new system compared to the former one, but if volatility is very high the consolidated budget will receive less under the new system compared to the former one (about 5% for projects with a little share of payroll $\mu \sim 0.2$ and 1% – 2% for those with a medium share $\mu \sim 0.5$). Under the new system, the NPV of the investor is always greater than under the former system (by about 30% for moderate volatility and by 5% – 7% for high volatility). Tax burden for a created firm under the new fiscal system will be slightly less than those under the former system: by about 10% to 15% for a moderate volatility project and by about 5% for high volatility projects.

Poorly technically rigged investment projects ($\psi \sim 0.2$). For those projects, depreciation allowances does not play a great role and a large share of payroll μ is typical. Under the new system the investor comes at about the same moment than with the former system (optimal levels of investment are very close). Tax revenue in the budget between the new and former systems are not significantly different (above all for high volatility projects). Tax burden for the enterprise is nearly the same (with the high degree of precision). The investor earns more profit in that case as in the preceding one, his NPV under the new system is always higher (by 10% – 15% for moderate volatility projects and by 3% for high volatility projects).

6.2. Estimation of the effect of the replacement of property tax by real estate tax

From 1997 in certain cities of Russia (especially in Tver and Novgorod), the real estate tax has been experimented. As previously mentioned, the experiment was established conformably to Federal Law from 20.06.1997 "About realization of the experiment under the taxation of the real estate in Great Novgorod and Tver cities". This law was initially adopted for three years and was extended for an additional period of three years in 2000.

The main idea of this experiment was to replace multiple taxes (property tax charged to enterprises, property tax charged to persons, land tax) by a unified real estate tax.

The base for this tax was determined as the market value of the real estate taxed at a flat rate for every real estate category[18].

When property tax is replaced by real estate tax, some elements of active assets (machinery, vehicles, equipment, stock, intangible assets and other property excluding buildings and constructions) which represent a large part of the base of the property tax, are no longer taxable. At the same time cost of the buildings and constructions can be far from the market value.

Such a taxation system based upon market value is applied in a growing number of countries and is favorable to the fiscal and stimulation function both in countries with a developed market economy and countries in transition. Real estate tax represents as much as 95% of local budget resources in the Netherlands, 81% in Canada, 52% in France. In the USA, depending of the State, this share represents from 10% to 70%. As reported by the World bank,

[18] Land with buildings and constructions located on this land are considered as an object of real estate

in a selection of developing countries studied, this tax represents 40% to 80% of the local budgets[19].

Concerning its *fiscal function*, the real estate tax provides a stable level of resources for local budgets as real estate is certainly one of the most stable and identifiable fiscal object. As far as the *stimulation function* is concerned, the fact that active assets are no longer taxed can stimulate the technical renewal of production and the shift to the market prices while calculating a tax base can lead to a more efficient use of the real estate resources.

As it will be shown later, in the framework of the proposed model, it is possible to evaluate the experiments of Tver and Novgorod from the point of view of investment activity in newly created enterprises.

Let us remark first that in the framework of our model this experiment means the replacement of property tax P_t^τ by

$$\tilde{P}_t^\tau = \tilde{\gamma}_p(1-\psi)I_t,$$

where $\tilde{\gamma}_p$ is the rate of real estate tax, and $(1-\psi)I_t$ is the value of inactive part of assets at the market price[20].

In this case, the formula of expected present value of the investor will be the following

$$\tilde{V}_\tau = \frac{(1-\mu)(1-\widehat{\gamma}_i)}{\rho+\delta-\alpha_2}\pi_\tau + I_\tau\left(H_1 - \frac{\tilde{\gamma}_p(1-\psi)}{\rho+\delta-\alpha_1}\tilde{H}_2\right),$$

where $\widehat{\gamma} i$ and H_1 are defined in (4.7), and

$$\tilde{H}_2 = 1-\gamma_i + \Delta\gamma_i\left[1-e^{-(\rho+\delta-\alpha_1)\nu}\right].$$

Optimal moment of investment in the conditions of this experiment will be

$$\tilde{\tau}^* = \min\{t \geq 0:\ \pi_t \geq \tilde{p}^* I_t\},$$

where $\tilde{p}^* = \left(1-H_1+\frac{\tilde{\gamma}_p(1-\psi)}{\rho+\delta-\alpha_1}\tilde{H}_2\right)\cdot\frac{\rho+\delta-\alpha_2}{(1-\mu)(1-\widehat{\gamma}_i)}\cdot\frac{\beta}{\beta-1}$ (cf (4.10)).

It is interesting to compare the optimal levels of investment for the current fiscal system (with property tax) and the experimental one (with real estate

[19] Data are taken from `http://www.valnet.ru`

[20] The price of land is not included here, for which a properly developed method of estimation does not exist yet

tax). Let us suppose for simplification purpose that tax holidays are absent ($\nu = 0$). In that case it is obtained from formulas cited above:

$$\tilde{R}_p = \frac{\tilde{p}^*}{p^*} = \frac{1 + \dfrac{\gamma_i}{1-\gamma_i}(1-K) + \dfrac{\tilde{\gamma}_p}{\rho+\delta-\alpha_1}(1-\psi)}{1 + \dfrac{\gamma_i}{1-\gamma_i}(1-K) + \dfrac{\gamma_p}{\rho+\delta-\theta\alpha_1}(1-K)},$$

where $K = \psi A_0 + (1-\psi)B_0$, and $A_0,\ B_0$ are defined in (4.5).

From then we can see that the change in investment activity is defined by the ratio between values $\dfrac{\tilde{\gamma}_p}{\rho+\delta-\alpha_1}(1-\psi)$ and $\dfrac{\gamma_p}{\rho+\delta-\theta\alpha_1}(1-K)$.

Therefore, in order for the investor to come earlier in the experimental system, (i.e. $\tilde{R}_p < 1$), it is necessary the following inequality holds:

$$\frac{\tilde{\gamma}_p}{\gamma_p}\cdot\frac{\rho+\delta-\theta\alpha_1}{\rho+\delta-\alpha_1} < \frac{1-K}{1-\psi} = 1 - B_0 + \frac{\psi}{1-\psi}(1-A_0).$$

This implies that in the experimental fiscal system the earlier arrival of the investor will happen on the case where active part of assets ψ will exceed a certain level $\widehat{\psi}$, equal to

$$\widehat{\psi} = \frac{B_0 + \dfrac{\tilde{\gamma}_p}{\gamma_p}\cdot\dfrac{\rho+\delta-\theta\alpha_1}{\rho+\delta-\alpha_1} - 1}{B_0 + \dfrac{\tilde{\gamma}_p}{\gamma_p}\cdot\dfrac{\rho+\delta-\theta\alpha_1}{\rho+\delta-\alpha_1} - A_0}.$$

In the following Table the calculations of threshold of the active part of assets $\widehat{\psi}$ are presented for different values of depreciation rates (λ for linear method – see (3.6), depreciation rate of inactive part of assets has been fixed to 3%) and of a risk parameter δ. In this case it has been supposed that the revaluation of active assets is absent ($\theta = 0$), discount rate has been fixed to $\rho = 10\%$, property and real estate taxes rates have been taken as established in the experiments ($\gamma_p = \tilde{\gamma}_p = 2\%$), the average rate of growth of investment resources (active assets) equaled $\alpha_1 = 1\%$.[21]

Thus, the replacement of the property tax by the real estate tax stimulates the coming of the investor only for properly technically rigged projects for wish the share of active assets in the initial investment exceeds a certain "critical" value. Under the absence of risk, this critical value is 60%–70% (a typical case), and it decreases along with the increase of the risk factor (for a value of the

[21] All values are presented on an annual basis

Table 6.3

δ	$\lambda = 15\%$	$\lambda = 25\%$
0%	0.60	0.69
5%	0.42	0.52
10%	0.31	0.39
15%	0.24	0.31

risk parameter of the same order as the discount, the critical share $\widehat{\psi}$ decreases by one half). This means that in a "risky" economy, the experimental fiscal system is able to stimulate the investment activity in a larger area (for the set of projects) than in an economy without risk.

6.3. Compensation of risk by means of tax mechanisms

When we described the model of investor there were two types of random factors influence on investor's profit from the project. The first one is connected with "market" fluctuations of profits around an average values and is specified by volatilities σ_1 and σ_2. These fluctuations can cause either a decrease in profit or a increase (according to changes in prices, demand etc.). The second type which is generated by social and legal environment leads to permanent declining in investor's cash flow. We will refer to those factors as risk (in proper sense). In the framework of our model the influence of such a risk is specified by average share of losses (in profit) per unit of time δ. Only this risk is a subject of our study in this Section.

In order to formulate a risk compensation problem let us consider the following hypothetical scheme. Suppose that the investor who follows the optimal behavior strategy, faces a dilemma: investing in a "risky" economy that provides significant tax exemptions; or going to "no-risk" economy and not to have any tax exemptions (or only minimal ones). There appears a question: which tax exemptions can compensate (from the investor's point of view) the risk factor.

We guess that the main criteria for the choice of one or another variant would be its expected net present value $\mathcal{N}$[22]. The exemptions may be connected with

[22] We will write also $\mathcal{N}(\cdot)$ in order to emphasize a dependence of NPV on the appropriate parameter

various tax mechanisms. Among such mechanisms we will pay our attention on the following: reduction in profit tax rate, changes in depreciation policy, and tax holidays (we shall notice that the role of the later is very limited in the latest tax reform). If we denote such a tax mechanism under risk δ as M_δ, then we call that mechanism M_δ *compensates* risk δ, if

$$\mathcal{N}(\delta, M_\delta) \geq \mathcal{N}(0, M_0). \tag{6.1}$$

In present report we focus on the first two mechanisms: compensation by means of profit tax rate reduction and compensation by means of depreciation policy. We will assume that tax holidays are absent ($\nu = 0$), and ignore (for simplification) assets tax (i.e. put $\gamma_p = 0$). Parameters of the project and discount rate will be assumed constant.

Compensation by means of profit tax rate

According to Theorem 2 the tax rate γ_i^δ compensates risk δ (in the sense of relation (6.1)) if and only if:

$$(1-\gamma_i^\delta K_\delta)^{1-\beta}\left(\frac{1-\gamma_i^\delta}{\rho+\delta-\alpha_2}\right)^\beta \geq (1-\gamma_i^0 K_0)^{1-\beta}\left(\frac{1-\gamma_i^0}{\rho-\alpha_2}\right)^\beta, \tag{6.2}$$

where γ_i^0 is profit tax rate in "no-risk" economy,

$K_\delta = \psi \int\limits_0^\infty a_t e^{-(\rho+\delta-\theta\alpha_1)t}dt + (1-\psi)\int\limits_0^\infty b_t e^{-(\rho+\delta-\theta\alpha_1)t}dt.$

(6.2) implies

$$\frac{1-\gamma_i^\delta}{(1-\gamma_i^\delta K_\delta)^{1-1/\beta}} \geq \frac{1-\gamma_i^0}{(1-\gamma_i^0 K_0)^{1-1/\beta}}\left(1+\frac{\delta}{\rho-\alpha_2}\right). \tag{6.3}$$

Since the left side of inequality (6.3) decreases in γ_i^δ (one can check it taking a derivative) and is 1 when $\gamma_i^\delta = 0$, then inequality (6.3) has a solution (in γ_i^δ) if and only if

$$\frac{1-\gamma_i^0}{(1-\gamma_i^0 K_0)^{1-1/\beta}}\left(1+\frac{\delta}{\rho-\alpha_2}\right) < 1, \quad \text{or} \quad \delta < \delta_0,$$

where $\delta_0 = \dfrac{\rho-\alpha_2}{1-\gamma_i^0}\left[(1-\gamma_i^0 K_0)^{1-1/\beta} - 1 + \gamma_i^0\right]$.

Thus, if risk exceeds the "critical" value δ_0, it cannot be compensated (from the investor's NPV point of view) by any reduction of profit tax rate. Taking into account non-negativity of $\gamma_i K_0$, we can give more crude, but more universal (depending on a smaller number of parameters) estimate $\delta_0 \le \bar{\delta}_0 = (\rho - \alpha_2)\dfrac{\gamma_i^0}{1-\gamma_i^0}$. If we take $\gamma_i^0 = 35\%$ as in most developed countries (which can be broadly considered as "no-risk"), then $\bar{\delta}_0 \approx 0.54(\rho - \alpha_2)$.

Compensation by means of depreciation policy

Assume now that we wish to compensate risk δ by means of depreciation schedule (a_t^δ) of an active part of the assets.

Let denote $K_\delta = \psi \int\limits_0^\infty a_t^\delta e^{-(\rho+\delta-\theta\alpha_1)t}dt + (1-\psi)\int\limits_0^\infty b_t e^{-(\rho+\delta-\theta\alpha_1)t}dt$ (depreciation of inactive part of assets is fixed).

In order to compensate risk δ (in the sense (6.1)) by depreciation (a_t^δ) the following inequality must be satisfied:

$$\frac{(1-\gamma_i^0 K_\delta)^{1-\beta}}{(\rho+\delta-\alpha_2)^\beta} \ge \frac{(1-\gamma_i^0 K_0)^{1-\beta}}{(\rho-\alpha_2)^\beta},$$

where depreciation schedule (a_t^0) (including in K_0) corresponds to "non-risky" case, or

$$1 + \frac{\delta}{\rho-\alpha_2} \le \left(\frac{1-\gamma_i^0 K_0}{1-\gamma_i^0 K_\delta}\right)^{1-1/\beta}. \tag{6.4}$$

Since for any depreciation schedule $K_\delta \le \psi + (1-\psi)B_0$, where $B_0 = \int\limits_0^\infty b_t e^{-(\rho+\delta-\theta\alpha_1)t}dt$, the inequality (6.4) holds if and only if

$$1 + \frac{\delta}{\rho-\alpha_2} < \left(\frac{1-\gamma_i^0 K_0}{1-\gamma_i^0[\psi+(1-\psi)B_0]}\right)^{1-1/\beta}, \quad \text{or } \delta < \delta_1,$$

where $\delta_1 = (\rho-\alpha_2)\left[\left(\dfrac{1-\gamma_i^0 K_0}{1-\gamma_i^0+\gamma_i^0(1-\psi)(1-B_0)]}\right)^{1-1/\beta} - 1\right]$.

Thus, like in the previous case (with profit tax rate), there is a "critical" value of risk δ_1, such that if risk is greater than this value, it can not be compensated by any depreciation policy.

It is interesting to compare the boundaries δ_0 and δ_1. It is easy to see that

$$\left\{1-\gamma_i^0+\gamma_i^0(1-\psi)(1-B_0)]\right\}^{1-1/\beta} > (1-\gamma_i^0)^{1-1/\beta} > 1-\gamma_i^0,$$

therefore, $\delta_1 < \delta_0$. This means that zone of risk which can be compensated by any choice of depreciation policy is less than zone of risk which can be compensated by a reduction of profit tax rate. In other words, depreciation mechanism has smaller opportunities for risk compensation than profit tax rate.

As many calculations (conducted on real adjusted data) have demonstrated, the frontier δ_0 is normally located on the interval 0.02 – 0.04, and the frontier δ_1 is smaller (even several times smaller). From this fact we can draw the conclusion that depreciation is clearly less able to compensate for risk than the profit tax rate. Apart from this, the possibilities to compensate for risk with the help of a reducing of the profit tax rate seem to be very limited.

6.4. The relation between depreciation rates for linear and non-linear methods

As we mentioned above, all depreciable assets are divided according to the new TC RF into ten groups depending on useful lifetime of the assets (Article 258). Assets from eighth to tenth groups (with useful lifetime more than 20 years) are depreciated by linear (straight-line) method, and from other groups can be depreciated either by linear or non-linear (declining balance) method according to enterprise's choice (Article 259). Besides there is the certain relation between rates of depreciation for linear and nonlinear methods, namely, nonlinear rate must be two times greater than the relevant linear rate (which is inverse to the useful lifetime). Similar relations exist in many other tax systems. It may be a fixed relation (as in Australia) or a scale depending on the useful lifetime of an asset (as in France or Spain)[23].

Let us suppose that tax holidays are absent. Then, as one can see from the formulas for calculating present value of the investor and present tax payments, the basic indicators (connected with enterprise activity) depend only on the present depreciation charges, i.e. on the integral of discounted depreciation schedule (DDS)

$$A = \int_0^\infty a_t e^{-\rho t}\, dt$$

This fact can be used in order to obtain the coefficients for the recalculation of depreciation rate for different methods.

As it is well known, the use of non-linear method implies a problem of "tail", since complete depreciation of assets with this method is impossible for any

[23] Data are taken from Cummins *et al.* (1996)

given finite interval of time. In Article 259 of the TC RF this problem is solved as follows. The non-linear method is used until when the residual cost of the assets attains 20% of its initial value, after that this residual cost is taken as a basis and it is depreciated using linear method before the end of the useful lifetime of assets.

In this Section we will show how, in the framework of the proposed model it is possible to obtain a relationship between depreciation rates under various methods.

We will show that the existing system of "switching" between non-linear method and linear method is not able to provide the invariance of DDS with none of the possible recalculation coefficients. At the same time, taking another "switching point" it is possible to provide the corresponding invariance.

So let L be the duration of useful life time of asset and q is the part of residual cost (to its initial value) such as when it is attained a switch happens from the non-linear to the linear method. In that case, DDS for the linear method of depreciation equals

$$A_{SL} = \frac{1 - e^{-\rho L}}{\rho L}. \tag{6.5}$$

If η is the rate of depreciation for the non-linear method, the moment of switch from the method T is defined by the following condition

$$R_T = e^{-\eta T} = q, \quad \text{or } \eta T = \ln q^{-1}.$$

For this method of depreciation DDS will be equal to

$$A_{DB} = \int_0^T \eta e^{-\eta t} e^{-\rho t}\, dt + \int_T^L \frac{R_T}{L - T} e^{-\rho t}\, dt.$$

If we take $\eta = k/L$, where $1/L$ is the rate of linear depreciation then it will be obtained

$$A_{DB} = \frac{k}{k + x}\left(1 - q e^{-yx}\right) + \frac{q}{x(1 - y)}\left(e^{-yx} - e^{-x}\right), \tag{6.6}$$

where $x = \rho L,\ y = (\ln q^{-1})/k$.

Hence, the "equivalent" recalculation coefficient k_0 has to be defined by the following equality

$$A_{SL} = A_{DB}. \tag{6.7}$$

From the formulas (6.5) and (6.7) follows that the coefficient k_0 must only be dependent on q and x. The main interest for us is the interval of value

$0.1 < x < 2$, on which the values ρL are located when those values of discount and the values of the duration of lifetime of assets (of the first to seventh groups of depreciation) are "reasonable". As it has been mentioned earlier, according to the current TC RF $q = 0.2$ and the coefficient of recalculation equals to $k = 2$. However, as conducted calculations show, in that case the following inequality will be held

$$A_{SL} < A_{DB}, \tag{6.8}$$

Let us note that the ratio A_{SL}/A_{DB} is varies from 0.99 for short lifetime, to 0.85 for long lifetime (from group 7). Moreover the inequality (6.8) remains valid for every possible values $k \geq \ln q^{-1}$ (for which $T \leq L$). This means that DDS calculated with linear method will always be less than DDS calculated with the non-linear method with the "tail".

Furthermore, if we change the switching point in the sense of an increase in q, we can obtain "equivalent" recalculation coefficient. As a variant we took $q = 0.3$, which means that the switching between the non-linear and linear method happens when residual cost attains the level of 30% of the initial cost.

In Table 6.4, we present recalculation coefficients that provide the equality (6.7) for various discount values and for existing groups of depreciated assets (for each group it has been taken the intermediate period of lifetime).

Table 6.4

Groups	Recalculation coefficient	
	$\rho = 10\%$	$\rho = 20\%$
1	1.48	1.47
2	1.47	1.46
3	1.46	1.44
4	1.45	1.42
5	1.44	1.39
6	1.42	1.35
7	1.39	1.29

As this Table shows, the recalculation coefficient for the 30%-switching scheme are located within the interval 1.4 – 1.5 and their dependence on the discount is very weak (above all for assets with short or moderate life time period).

The non-linear depreciation scheme with 20%-switching point (currently in place) is unable to provide firms with an equivalent choice between linear and non-linear methods for any recalculation coefficient between the rates. The relative difference for the depreciation deductions listed above for the two methods with a recalculation coefficient equal to 2 established by the TC RF fluctuates between 1% – 2% for short life time period and 15% – 20% for sufficiently long life time period of group 7. But for the non-linear scheme with 30%-switching point there exist recalculation coefficients of depreciation rates which lead to an "equivalent" choice between linear and non-linear methods. Those "equivalent" coefficients which depend on the discount and on the useful life time of depreciated assets, are located (for "reasonable" values of the parameters) within the interval 1.4 – 1.5, and are weakly sensible to the change in discount.

6.5. Losses carry forward: various schemes

As we noted above, when we described the scheme of profit taxation, we made a simplification. It concerns the case when costs exceed incomes and hence firm incurs losses.

From the formula (3.10) for the present value of the firm one can see that, in the case of negative profits, present tax payments are decrease also (proportionally to the profit) with discounting. Practically, it means that losses are fully (with interest) reimburse.

But this principle is too favorable for a firm. Actually, a majority of tax systems allows to carry forward losses (i.e. to summarize losses and future profits) without interest.

According to TC RF, tax base for negative profit is zero, and losses are allowed to carry forward as deductions from tax base. Such deductions are subject to some restrictions: first, they are limited to a certain time interval (10 years); and, second, they can not reduce tax base by more than 30%.

Unfortunately, this actual scheme is enough difficult for investigation. Most of all, it does not concern numeric calculations, but a study of the dependencies of indicators of investment activity on different parameters. From this point of view the proposed scheme of losses' "full reimbursement", seems to be valuable enough both for analysis and calculations. We will follow this scheme in the future.

In order to understand how much this scheme is far from an actual one, we describe below a model of "actual reimbursement" scheme and then we give estimates for comparison of these two schemes.

1. For simplicity we will consider deterministic processes of value added and required investments. It is assumed also that tax holidays are absent ($\nu = 0$), and investment moment equals τ.

Let firm's profit at the moment t after investing ($t > \tau$) be $Z_t = Q_t - D_t$, where Q_t be value added minus payroll fund, both assets and social taxes, and D_t be the depreciation charges[24]. Suppose, that the profit is negative for some initial period after investing (for $\tau \le t < \tau + t_0$), and positive for $t \ge \tau + t_0$. We will also ignore temporal limitations for losses carry forward and consider that this carry forward can be performed for unbounded time interval[25].

For this profit the levied tax according to TC RF is the following:

$$\tilde{T}_t = \begin{cases} 0, & \text{if } \tau \le t \le \tau + t_0 \\ \gamma_i (Z_t - l_t), & \text{if } \tau + t_0 < t \le \tau + \tilde{t}_1 \\ \gamma_i Z_t, & \text{if } t > \tau + \tilde{t}_1 \end{cases},$$

where l_t is the flow of deductions (on loss reimbursement) continued up to the moment when all accumulated losses (that is $\int\limits_{\tau}^{\tau+t_0} (-Z_t)\, dt$) will be compensated. Thus, the moment $\tilde{t}_1$, which specifies ending the deductions, can be found from the relation

$$\int\limits_{\tau+t_0}^{\tau+\tilde{t}_1} l_t\, dt = - \int\limits_{\tau}^{\tau+t_0} Z_t\, dt.$$

Since deductions must not decrease tax base by more than the share φ $(0 \le \varphi \le 1)$, then it is natural to put $l_t = \varphi Z_t$. Therefore, the moment $\tilde{t}_1$ is defined from the relation

$$\int\limits_{\tau}^{\tau+t_0} Z_t\, dt + \varphi \int\limits_{\tau+t_0}^{\tau+\tilde{t}_1} Z_t\, dt = 0. \tag{6.9}$$

The net present value of the firm, derived according the proposed "actual reimbursement" scheme is

[24] Here we will skip index τ which specifies an investment moment

[25] Practice of unbounded losses carry forward takes place in some countries (for example, in Germany, Great Britain, Chile, Estonia – see *Problems of Tax System in Russia* ..., 2000)

$$\begin{aligned}\tilde{V}_\tau &= \int\limits_{\tau}^{\infty}(Q_t - \tilde{T}_t)e^{-\rho(t-\tau)}\,dt = \int\limits_{\tau}^{\tau+t_0} Q_t e^{-\rho(t-\tau)}\,dt \\ &+ \int\limits_{\tau+t_0}^{\tau+\tilde{t}_1}[Q_t - \gamma_i(1-\varphi)Z_t]e^{-\rho(t-\tau)}\,dt + \int\limits_{\tau+\tilde{t}_1}^{\infty}[(1-\gamma_i)Q_t + \gamma_i D_t]e^{-\rho(t-\tau)}\,dt.\end{aligned}$$

The difference with the net present value of the firm, derived according the "full reimbursement" scheme (formula (3.10)) equals

$$\begin{aligned}\tilde{V}_\tau - V_\tau &= \int\limits_{\tau}^{\tau+t_0}[Q_t - (1-\gamma_i)Q_t - \gamma_i D_t]e^{-\rho(t-\tau)}\,dt \\ &+ \int\limits_{\tau+t_0}^{\tau+\tilde{t}_1}[Q_t - \gamma_i(1-\varphi)Z_t - (1-\gamma_i)Q_t - \gamma_i D_t]e^{-\rho(t-\tau)}\,dt \\ &= \gamma_i\left[\int\limits_{\tau}^{\tau+t_0} Z_t e^{-\rho(t-\tau)}\,dt + \varphi\int\limits_{\tau+t_0}^{\tau+\tilde{t}_1} Z_t e^{-\rho(t-\tau)}\,dt\right]. \qquad (6.10)\end{aligned}$$

Using the negativity and positivity of Z_t at correspondent intervals, we have

$$\int\limits_{\tau}^{\tau+t_0} Z_t e^{-\rho(t-\tau)}\,dt \le e^{-\rho t_0}\int\limits_{\tau}^{\tau+t_0} Z_t\,dt, \quad \int\limits_{\tau+t_0}^{\tau+\tilde{t}_1} Z_t e^{-\rho(t-\tau)}\,dt \le e^{-\rho t_0}\int\limits_{\tau+t_0}^{\tau+\tilde{t}_1} Z_t\,dt.$$

Further, (6.10) implies

$$\tilde{V}_\tau - V_\tau \le \gamma_i e^{-\rho t_0}\left[\int\limits_{\tau}^{\tau+t_0} Z_t\,dt + \varphi\int\limits_{\tau+t_0}^{\tau+\tilde{t}_1} Z_t\,dt\right] = 0$$

due to condition (6.9).

Therefore, NPV for the "actual reimbursement" scheme is always less (no greater) than NPV for the "full reimbursement" scheme.

Note, that if we determine the moment of deductions ending $\bar{t}_1$ not as in (6.9), but from the equality of discounted losses and discounted deductions, i.e. by the relation

$$\int_{\tau+t_0}^{\tau+\bar{t}_1} l_t e^{-\rho(t-\tau)}\, dt = -\int_{\tau}^{\tau+t_0} Z_t e^{-\rho(t-\tau)}\, dt,$$

then (6.10) implies that the relevant $\tilde{V}_\tau$ is the same as V_τ. This means that the scheme proposed in our model is equivalent to the scheme with limited deduction where the ending deductions $\bar{t}_1$ determines from the principle of "discounted reimbursement". It is easy to see that the moment $\bar{t}_1$ is always greater than the analogous moment $\tilde{t}_1$, specified by (6.9).

2. Let us pass to an estimating rate of approximation of one scheme by another.

In order to avoid too complicated formulas, we will assume that the amount of required investments is time-independent (i.e. $I_t \equiv I$), the value added π_t grows exponentially with the rate $\alpha_2 = \alpha$, and not take into account the property tax (i.e. put $\gamma_p = 0$). Then $Q_t = (1-\mu)\pi_t$ is the value added minus payroll fund and social tax. Furthermore, we will gather all assets in one common group and will depreciate its by the single manner (by depreciation schedule $(a_t,\ t \geq 0)$).

We have:

$$\begin{aligned}
\int_{\tau}^{\tau+t_0} Z_t e^{-\rho(t-\tau)}\, dt &= \int_0^{t_0} Q_{\tau+t} e^{-\rho t}\, dt - I\int_0^{t_0} a_t e^{-\rho t}\, dt \\
&= \frac{1-\mu}{\widehat{\rho}}\pi_\tau \left(1 - e^{-\widehat{\rho} t_0}\right) - I(A_0 - A_{t_0}), \\
\int_{\tau+t_0}^{\tau+\tilde{t}_1} Z_t e^{-\rho(t-\tau)}\, dt &= \int_{t_0}^{\tilde{t}_1} \tilde{\pi}_{\tau+t} e^{-\rho t}\, dt - I\int_{t_0}^{\tilde{t}_1} a_t e^{-\rho t}\, dt \\
&= \frac{1-\mu}{\widehat{\rho}}\pi_\tau \left(e^{-\widehat{\rho} t_0} - e^{-\widehat{\rho}\tilde{t}_1}\right) - I(A_{t_0} - A_{t_1}),
\end{aligned}$$

where $\widehat{\rho} = \rho - \alpha, \quad A_t = \int_t^\infty a_s e^{-\rho s}\, ds$. Then (6.10) implies

$$\tilde{V}_\tau - V_\tau = \frac{1-\mu}{\widehat{\rho}}\pi_\tau \left(1-(1-\varphi)e^{-\widehat{\rho} t_0} - \varphi e^{-\widehat{\rho}\tilde{t}_1}\right) - I\left[A_0 - (1-\varphi)A_{t_0} - \varphi A_{\tilde{t}_1}\right],$$

and, taking into account that $V_\tau = \dfrac{(1-\mu)(1-\gamma_i)}{\widehat{\rho}}\pi_\tau + \gamma_i I A_0$ (see formula (4.8)), we have the following relative estimate of an approximation of "actual reimbursement" scheme by the proposed "full reimbursement" scheme:

$$R_\tau := \frac{\tilde{V}_\tau - V_\tau}{V_\tau} = \frac{p_\tau(1-\mu)\tilde{E} - \widehat{\rho}\tilde{A}}{p_\tau(1-\mu) + \widehat{\rho}\gamma_i A_0},$$

where $p_\tau{=}\pi_\tau/I, \quad \tilde{E}{=}1{-}(1{-}\varphi)e^{-\widehat{\rho}t_0}{-}\varphi e^{-\widehat{\rho}\tilde{t}_1}, \quad \tilde{A}{=}A_0{-}(1{-}\varphi)A_{t_0}{-}\varphi A_{\tilde{t}_1}$. Taking the optimal investment moment τ^* from Theorem 1, we obtain

$$R_{\tau^*} = \frac{\gamma_i}{\rho - \alpha\gamma_i A_0}\left(\rho\frac{1-\gamma_i A_0}{1-\gamma_i}\tilde{E} - (\rho-\alpha)\tilde{A}\right) \tag{6.11}$$

The derived estimation (6.11) depends on $t_0 \;\; \tilde{t}_1$, which specify a period of tax deductions. As one can see the moment t_0 can be specified as

$$t_0 = \max\{t \geq 0 : \; Q_{t+\tau} \leq D_t\}, \tag{6.12}$$

(here we suppose, of course, that $Q_\tau \leq D_0$, else profit will be positive forever), and $\tilde{t}_1$ is a root of the equation

$$g_1(t) = g_2(t), \tag{6.13}$$

where

$$g_1(t){=}(1{-}\varphi)\int_0^{t_0} D_s\,ds{+}\varphi\int_0^{t} D_s\,ds, \quad g_2(t){=}(1{-}\varphi)\int_0^{t_0} Q_{\tau+s}\,ds{+}\varphi\int_0^{t} Q_{\tau+s}\,ds.$$

Further calculation of the values t_0 and $\tilde{t}_1$ in (6.12) and (6.13) depends on concrete depreciation methods. We have carried out numerous calculations both for linear and non-linear methods. As model's parameters we took $\alpha \sim 1\% - 2\%,\; \rho \sim 10\% - 15\%$ (all are on annual basis), tax rate $\gamma_i = 24\%$, maximal share of taxable profit deductions $\varphi = 30\%$ (as the new TC RF establishes). Those calculations showed that for "intermediate" depreciation rates (related to the "intermediate" useful lifetimes of assets 5–10 years) the relative approximation rate R_{τ^*} from a replacement of actual reimbursement scheme by the proposed full reimbursement scheme does not exceed 5%.

Finally note that analogous estimates can be derived also for the discounted tax payments into the budget.

7. MATHEMATICAL APPENDIX

In this Chapter, will be described the mathematical tools, which will be used to study the proposed model. We will deal with an optimal stopping problem for multi-dimensional stochastic process. This problem appears in determination of an optimal moment for investment (3.11).

We propose a new approach to find the optimal stopping time for multi-dimensional diffusion processes. On the basis of this approach an explicit formula for the optimal investment moment in the case of two-dimensional geometric Brownian motion and homogeneous objective functional will be obtained[26].

7.1. The optimal stopping time for multi-dimensional diffusion process

Let $(\xi_t = (\xi_t^1, \dots, \xi_t^m),\ t \geq 0)$ be multi-dimensional diffusion process with values in R^m, which is described by the following system of stochastic differential equations:

$$d\xi_t^i = a_i(\xi_t)dt + \sum_{k=1}^{m} b_{ki}(\xi_t)dw_t^k, \quad (t \geq 0),\ i = 1, \dots, m, \quad \xi_0 = x^0 \quad (7.1)$$

where $a = (a_1, \dots, a_m),\ b_k = (b_{k1}, \dots, b_{km})$ are continuous vector functions on R^m, $(w_t^k,\ t \geq 0),\ k = 1, \dots, m$ are independent Wiener processes, x^0 is the initial state.

Let us consider an optimal stopping problem for this process:

$$\mathbf{E}e^{-\rho\tau} g(\xi_\tau) \to \max_{\tau}, \quad (7.2)$$

where $g : R^m \to R^1$, and the maximum is taken over a certain class of Markov times τ (where the class of all Markov times $\mathcal{M}$ is usually examined).

Though the general theory of an optimal stopping is well developed (see, for example, Shiryaev, 1978; Oksendal, 1998), there are very few problems which have solutions in an explicit form. The traditional approach to solving a

[26] The results of this Chapter were presented at the Conference on Stochastic dynamic optimization (Vienna, 2002) and the World Congress of Bachelier Finance Society (Crete, 2002)

problem (7.2) is the heuristic method of "smooth pasting" (or "high contact") for the differential equations with free boundary.

Let L be the generator of process (7.1), defined on functions from $C^2(R^m)$, which is given by the formula:

$$Lf(x) = \sum_{i=1}^{m} a_i(x) f'_{x_i} + \frac{1}{2} \sum_{i,j=1}^{m} \sum_{k=1}^{m} b_{ki}(x) b_{kj}(x) f''_{x_i x_j}.$$

In order not to consider degenerative (deterministic) processes, we will assume that L is an elliptic operator, i.e. eigenvalues of a diffusion matrix $\| \sum_{k=1}^{m} b_{ki}(x) b_{kj}(x) \|$ are positive.

Let $F(x)$ be the optimal functional value in problem (7.2), and $G=\{x \in R^m : g(x) < F(x)\}$ is "continuation region". Then $F(x)$ as a function of initial state of process $\xi_0 = x$, satisfies the differential equation $LF(x) = \rho x$ in G and "continuous pasting" condition $F(x) = g(x)$ at the boundary ∂G.

The specific feature of this problem is that the region G is unknown and is a subject of the search. In order to find this region, some additional conditions are usually involved, which are connected with equality of derivatives of functions $F(x)$ and $g(x)$ ("smooth pasting") on the boundary ∂G[27]. The general theory offers some sufficient conditions, for which the solution received by "smooth pasting" method, will be optimal (see, for example, Shiryaev, 1978). Unfortunately, these conditions are not practically checked. Therefore "smooth pasting" method is considered for concrete optimal stopping problems as only heuristic method of the solution finding, whose optimality requires additional proof[28].

7.2. Variational approach to optimal stopping problems

We propose the other approach to solve an optimal stopping problem (7.2). It is known from the general theory, that the optimal stopping time can be presented as the first exit time from some "continuation region" for diffusion process (see, for example, Shiryaev, 1978; Oksendal, 1998). The approach below is based on a variation of "continuation region" over the given class of regions.

[27] Sometimes it is considered partial derivatives or derivatives in direction. The variants of "smooth pasting" conditions one can find in Shiryaev (1978), Oksendal (1998). Such problems for differential equations with free boundary are called Stefan's problems

[28] Let us note, that the "smooth pasting" method can give in some cases both "maximal" and "minimal" solution, or no solution at all. In this sense all the solutions of optimal stopping problem from Dixit and Pindyck (1994), McDonald and Siegel (1986), Trigeorgis (1996) are not rigorous

Let $\mathcal{G}$ be some class of "continuation regions" in R^m, $G \in \mathcal{G}$.

Let $\tau_G = \tau_G(x) = \min\{t \geq 0 : \ \xi_t \notin G\}$ be the first exit time from region G for the process ξ_t, described by the equations (7.1) with initial state $\xi_0 = x$. Let $\mathcal{M}(\mathcal{G}) = \{\tau_G, \ G \in \mathcal{G}\}$ be a set of the first exit times for all regions from class $\mathcal{G}$.

It is known, that under some assumptions the function

$$u(x) = \mathbf{E}e^{-\rho\tau_G} g(\xi_{\tau_G})$$

is the solution of Dirichlet boundary problem:

$$\begin{aligned} Lu(x) &= \rho u(x), \quad x \in G, & (7.3)\\ u(x) &\to g(a), \quad \text{when } x \to a, \ x \in D, \ a \in \partial D. & (7.4) \end{aligned}$$

(Variants of this statement that sometime named as the Feynmann-Kac formula, under various assumptions one can find in Karatzas and Shreve, 1991; Krylov, 1996; Oksendal, 1998.)

Fix the initial state of process (7.1) $\xi_0 = x^0$. Then for each continuation region $G \subset \mathcal{G}$ we will consider the solution of the problem (7.3)–(7.4) $u_G(x^0)$ as a functional on set of regions $\mathcal{G}$.

Thus, a solving an optimal stopping problem (7.2) over a class of the Markov times $\mathcal{M}(\mathcal{G})$ can be reduced to a solving the following variational problem:

$$u_G(x^0) \to \max_{G \in \mathcal{G}}. \tag{7.5}$$

If G^* is an optimal region in (7.5), the optimal stopping time in the class $\mathcal{M}(\mathcal{G})$ coincides with the first exit time from this region: $\tau^*(\mathcal{G}) = \tau_{G^*}$. If the class of regions $\mathcal{G}$ is chosen "well", it is possible to prove that the moment $\tau^*(\mathcal{G})$ will be also an optimal stopping time for problem (7.2) over all Markov times $\mathcal{M}$. In the following Section such approach will be realized for two-dimension geometric Brownian motion ξ_t and for homogeneous objective function g.

Let us note, that the calculation of the optimal stopping time over a given class of regions represents, to our opinion, a peculiar interest. Indeed, the optimum continuation region for multi-dimensional diffusion processes can have very complex structure, therefore it has a sense to restrict our consideration to more simple regions, thus solution for problems (7.3)–(7.4) and (7.5) can be obtained by numerical methods.

7.3. Two-dimensional geometric Brownian motion

We will apply the proposed approach to solve an optimal stopping problem for the case of two-dimensional geometric Brownian motion.

Let us consider the following two-dimensional diffusion process $\xi_t = (\xi_t^1, \xi_t^2)$, $t \geq 0$, which will described the processes of value added and amount of investment:

$$\begin{array}{ll} d\xi_t^1 = \xi_t^1(\alpha_1 dt + \sigma_1 d\tilde{w}_t^1), & \xi_0^1 = x_1, \\ d\xi_t^2 = \xi_t^2(\alpha_2 dt + \sigma_2 d\tilde{w}_t^2), & \xi_0^2 = x_2, \end{array} \tag{7.6}$$

where pair $(\tilde{w}_t^1, \tilde{w}_t^2)$ is two-dimensional Wiener process with correlated components: $\mathbf{E}\tilde{w}_t^1\tilde{w}_t^2 = rt, (|r| \leq 1)$.

In order to reduce this process to the canonical form (7.1), we introduce new Wiener processes

$$w_t^1 = \tilde{w}_t^1, \qquad w_t^2 = (\tilde{w}_t^2 - r\tilde{w}_t^1)/\sqrt{1-r^2} \quad (\text{ if } |r| < 1).$$

As one can see, the processes w_t^1 and w_t^2 are uncorrelated ($\mathbf{E}w_t^1 w_t^2 = 0$) and, thus, are independent. Therefore, the process ξ_t can be presented in the following form:

$$\begin{array}{l} d\xi_t^1 = \xi_t^1(\alpha_1 dt + \sigma_1 dw_t^1), \\ d\xi_t^2 = \xi_t^2[\alpha_2 dt + \sigma_2(r dw_t^1 + \sqrt{1-r^2} dw_t^2)], \end{array} \tag{7.7}$$

where w_t^1 and w_t^2 are independent Wiener processes.

As continuation regions in R_+^2 we will consider a family of sets depending on the parameter p of the following type

$$G_p = \{(x_1, x_2) \in R_+^2 : \ x_2 < p x_1\}, \quad p \geq 0.$$

For the process $\xi = (\xi_t^1, \xi_t^2)$, described by the system of equations (7.6) with initial state $x = (x_1, x_2) \in R_+^2$, let us denote $\tau_p(x) = \min\{t \geq 0 : \ \xi_t \notin G_p\} = \min\{t \geq 0 : \ \xi_t^2 \geq p\xi_t^1\}$ – the first exit time from region G_p.

Let us consider the following functional

$$F_p(x) = \mathbf{E}^x e^{-\rho\tau_p(x)} g(\xi_{\tau_p(x)}), \quad x \in R_+^2$$

(the upper index x at the expectation symbol emphasizes, that the process ξ_t starts from the point x). If $x \notin G_p$, then $\tau_p(x) = 0$ and, hence, $F_p(x) = g(x)$.

Let us remind, that the function $g : R^2_+ \to R^1$ is called a homogeneous (of degree q) function, if

$$g(\lambda x) = \lambda^q g(x) \quad \text{for all } x \in R^2_+ \text{ and } \quad \lambda \geq 0.$$

Let us denote $\tilde{\sigma}^2 = \sigma_1^2 - 2r\sigma_1\sigma_2 + \sigma_2^2$ – "total" volatility of the process (7.6), and assume that $\tilde{\sigma} > 0$.

Theorem 4. *Let function $g(x)$ be a homogeneous (of degree q) function. Let $\bar{\alpha}_i = \alpha_i + \frac{q-1}{2}\sigma_i^2 \quad (i = 1, 2)$. Suppose the following conditions hold:*

$$\alpha_2 - \frac{1}{2}\sigma_2^2 \geq \alpha_1 - \frac{1}{2}\sigma_1^2, \tag{7.8}$$

$$\rho > \bar{\alpha}_1 q. \tag{7.9}$$

Then:

$$F_p(x_1, x_2) = \begin{cases} g(1,p)p^{-\beta} x_1^{q-\beta} x_2^{\beta}, & \text{if } x_2 < px_1 \\ g(x_1, x_2), & \text{if } x_2 \geq px_1 \end{cases},$$

where β is the positive root of the quadratic equation

$$\frac{1}{2}\tilde{\sigma}^2\beta(\beta - 1) + (\bar{\alpha}_2 - \bar{\alpha}_1 - \frac{q-1}{2}\tilde{\sigma}^2)\beta - (\rho - \bar{\alpha}_1 q) = 0. \tag{7.10}$$

Further we will use this theorem for the cases of the unit and zero degrees of homogeneity.

The variation problem (7.5) in the considered case has the following form:

$$F_p(x^0) \to \max_{p \geq 0}. \tag{7.11}$$

The explicit form of the functional F_p from Theorem 1 allows us to find the solution for the problem (7.11) and, therefore, the solution of optimal stopping problem (7.2) over the class of Markov times $\mathcal{M}(\mathcal{G}_0)$, where $\mathcal{G}_0 = \{G_p,\ p \geq 0\}$.

Let us denote $h(p) = g(1, p)p^{-\beta} \quad (0 \leq p < \infty)$.

Theorem 5. *Let the conditions of Theorem 4 hold, and p^* be a point of maximum of function $h(p)$. Then :*

1) the maximum in problem (7.11) is attained if $p = p^*$ (and thus does not depend on x^0);

2) optimal stopping time for problem (7.2) over the class $\mathcal{M}(\mathcal{G}_0)$ *is* $\tau^* = \min\{t \geq 0: \ \xi_t^2 \geq p^* \xi_t^1\}$,
3) optimal value of the functional for problem (7.2) over the class $\mathcal{M}(\mathcal{G}_0)$, *depending on initial state* (x_1, x_2) *of the process (7.6) is*

$$\Phi(x_1, x_2) = \begin{cases} h(p^*) x_1^{q-\beta} x_2^{\beta}, & \text{if } x_2 < p^* x_1 \\ g(x_1, x_2), & \text{if } x_2 \geq p^* x_1 \end{cases} . \tag{7.12}$$

It is turned out that under some additional conditions the set G_{p^*} determines also an optimal stopping time for problem (7.2) over the class of all Markov times $\mathcal{M}$.

Theorem 6. *Let the conditions of Theorem 4 hold,* $g \in C^2(R_+^2)$, p^* *be a point of strict maximum of function* $h(p)$[29], *and the following relations are satisfied for all* $p \geq p^*$*:*

$$h'(p) \leq 0, \tag{7.13}$$

$$p g''_{x_2 x_2}(1, p) - (\beta - 1) g'_{x_2}(1, p) \leq 0. \tag{7.14}$$

Then $\tau^* = \min\{t \geq 0: \ \xi_t^2 \geq p^* \xi_t^1\}$ *is the optimal stopping time for problem (7.2) over all Markov times* $\mathcal{M}$, *and the function (7.12) is the optimal value of the functional in (7.2) depending on initial state* (x_1, x_2) *of the process (7.6).*

For the proof of this theorem we use the sufficient conditions of optimality for the stopping time based on a method of variational inequalities (see, for example, Bensoussan and Lions, 1987; Oksendal, 1998).

Let us consider a corollary of this theorem for the linear function $g(x_1, x_2) = x_2 - x_1$. This case arises in a problem of the optimal timing for investment.

Corollary. *Let* $g(x_1, x_2) = x_2 - x_1$, *condition (7.8) hold, and* $\rho > \max(\alpha_1, \alpha_2)$. *Then* $\tau^* = \min\{t \geq 0: \ \xi_t^2 \geq p^* \xi_t^1\}$, *where* $p^* = \beta/(\beta - 1)$, *and* β *is the positive root of the quadratic equation*

$$\frac{1}{2}\tilde{\sigma}^2 \beta(\beta - 1) + (\alpha_2 - \alpha_1)\beta - (\rho - \alpha_1) = 0.$$

The formula for the optimal stopping time for a difference of two geometric Brownian motions was first given (from heuristic arguments) in McDonald and Siegel (1986). The strict proof of an optimality, and the conditions at which this formula is hold, appeared later (Hu and Oksendal, 1998).

[29] I.e. $h(p^*) > h(p)$ for $p \neq p^*$

Let us notice in conclusion, that if maximum point $p^* > 0$, then the necessary condition of optimality is the following

$$h'(p^*) = 0, \quad \text{or} \;\; p^* g'_{x_2}(1, p^*) = \beta g(1, p^*),$$

and it coincides with a "smooth pasting" condition for optimal value of the functional (7.12):

$$\Phi'_{x_2}(x_1, p^* x_1 - 0) = g'_{x_2}(x_1, p^* x_1).$$

Thus, for two-dimensional geometric Brownian motion and homogeneous objective function "smooth pasting" conditions follows from Theorem 5.

7.4. Proofs

The proof of Theorem 4. We will need the following

Lemma. *If condition (7.10) holds, then $\tau_p(x) < \infty$ (a.s.) for any $x \in R^2_+$ and $p > 0$.*

The proof. From the explicit representation for (one-dimensional) geometric Brownian motion in (7.6) we have:

$$\frac{\xi_t^2}{\xi_t^1} = \frac{x_2}{x_1} \exp\{(\tilde{\alpha}_2 - \tilde{\alpha}_1)t + \sigma_2 \tilde{w}_t^2 - \sigma_1 \tilde{w}_t^1\}, \tag{7.15}$$

where $\tilde{\alpha}_i = \alpha_i - \frac{1}{2}\sigma_i^2 \quad (i = 1, 2)$. According to the law of iterated logarithm for Wiener process (see Karatzas and Shreve, 1991)

$$\limsup_{t \to \infty} |\tilde{w}_t^i| / \sqrt{2t \log \log t} = 1 \quad \text{a.s.} \qquad (i = 1, 2).$$

Therefore, (7.15) implies that under $\tilde{\alpha}_2 \geq \tilde{\alpha}_1$

$$\limsup_{t \to \infty} \xi_t^2 / \xi_t^1 = \infty \qquad \text{a.s.}$$

Hence, $\tau_p(x) = \min\{t \geq 0 : \xi_t^2/\xi_t^1 \geq p\} < \infty$ for any $x \in R^2_+$ and $p > 0$ under (7.8).

Return to the proof of Theorem 4. First of all show that $F_p(x)$ is a homogeneous (of degree q) function.

Since $\tau_p(x)$ is a first exit time over the level p of a process ξ_t^2/ξ_t^1, then formula (7.15) implies that function $\tau_p(x)$ is homogeneous (of zero degree) in $x = (x_1, x_2)$.

Furthermore, the linear homogeneity of process ξ_t in initial point and homogeneity of function g imply:

$$\begin{aligned} F_p(\lambda x) &= \mathbf{E}^{\lambda x} e^{-\rho\tau_p(\lambda x)} g(\xi_{\tau_p(\lambda x)}) = \mathbf{E}^{\lambda x} e^{-\rho\tau_p(x)} g(\xi_{\tau_p(x)}) \\ &= \mathbf{E}^{x} e^{-\rho\tau_p(x)} g(\lambda\xi_{\tau_p(x)}) = \lambda^q F_p(x), \end{aligned}$$

i.e. $F_p(x)$ is homogeneous (of degree q) function.

Let us find $F_p(x)$ as a solution of Dirichlet problem (7.3)–(7.4). Homogeneous function $F_p(x)$ can be represented as

$$F_p(x_1, x_2) = x_1^q f(y), \quad \text{where } y = \frac{x_2}{x_1}, \quad f(y) = F_p(1, y).$$

Then

$$\begin{aligned} y'_{x_1} &= -\frac{y}{x_1}, \quad y'_{x_2} = -\frac{1}{x_1}, \\ F'_{x_1} &= q x_1^{q-1} f(y) + x_1^q f'(y)\left(-\frac{y}{x_1}\right) = x_1^{q-1}[qf(y) - yf'(y)], \\ F'_{x_2} &= x_1^{q-1} f'(y), \\ F''_{x_2x_2} &= x_1^{q-2} f''(y), \\ F''_{x_1x_2} &= (q-1)x_1^{q-2} f'(y) + x_1^{q-1} f''(y)\left(-\frac{y}{x_1}\right) = x_1^{q-2}[(q-1)f'(y) - yf''(y)], \\ F''_{x_1x_1} &= (q-1)x_1^{q-2}[qf(y) - yf'(y)] + x_1^{q-1}\left[qf'(y)\left(-\frac{y}{x_1}\right) + \frac{y}{x_1}f'(y)\right. \\ &\quad \left. - yf''(y)\left(-\frac{y}{x_1}\right)\right] = x_1^{q-2}\left\{(q-1)[qf(y) - yf'(y)]\right. \\ &\quad \left. - y[(q-1)f'(y) - yf''(y)]\right\} = x_1^{q-2}\{(q-1)[qf(y) - 2yf'(y)] + y^2 f''(y)\}. \end{aligned}$$

From the relation (7.7) we have that the generator of the process ξ_t is

$$LF(x) = \alpha_1 x_1 F'_{x_1} + \alpha_2 x_2 F'_{x_2} + \frac{1}{2}\sigma_1^2 x_1^2 F''_{x_1x_1} + r\sigma_1\sigma_2 x_1 x_2 F''_{x_1x_2} + \frac{1}{2}\sigma_2^2 x_2^2 F''_{x_2x_2}. \tag{7.16}$$

Equation (7.3) and formula (7.16) for the elliptic operator L lead to the following relation

$$\begin{aligned} \rho f(y) &= \alpha_1[qf(y) - yf'(y)] + \alpha_2 y f'(y) + \frac{1}{2}\sigma_1^2\{(q-1)[qf(y) - 2yf'(y)] + y^2 f''(y)\} \\ &\quad + r\sigma_1\sigma_2 y[(q-1)f'(y) - yf''(y)] + \frac{1}{2}\sigma_2^2 y^2 f''(y), \end{aligned}$$

or

$$\frac{1}{2}y^2 f''(y)\tilde{\sigma}^2 + yf'(y)[\bar{\alpha}_2 - \bar{\alpha}_1 - \frac{q-1}{2}\tilde{\sigma}^2] - f(y)(\rho - \bar{\alpha}_1 q) = 0. \quad (7.17)$$

The solution of second-order homogeneous differential equation (respect to function $f(y)$) (7.17) will be found in the type $f(y) = Cy^\beta$, where C is a constant. In this case β must be a root of the quadratic equation (7.10).

If (7.9) is satisfied, the equation (7.10) has two roots: positive β_1 and negative β_2. Hence, any solution of equation (7.17) for $0 < y < p$ has the following type:

$$f(y) = C_1 y^{\beta_1} + C_2 y^{\beta_2}, \quad \text{where } \beta_1 > 0,\ \beta_2 < 0,$$

or, returning to initial function,

$$F_p(x_1, x_2) = C_1 x_1^{q-\beta_1} x_2^{\beta_1} + C_2 x_1^{q-\beta_2} x_2^{\beta_2}, \quad \text{if } 0 < x_2 \le px_1, x_1 > 0 \quad (7.18)$$

By the condition (7.4), if $x_2 \to 0$ and $x_2 < px_1$ then $F_p(x_1, x_2) \to g(x_1, 0)$, therefore $C_2 = 0$ in representation (7.18). C_1 is found from the boundary condition at the line $\{x_2 = px_1\}$, namely,

$$F_p(x_1, px_1) = C_1 x_1^q p^{\beta_1} = g(x_1, px_1) = x_1^q g(1, p),$$

i.e. $C_1 = g(1, p)p^{-\beta}$.

Theorem is proved.

The proof of Theorem 5. Let us take any $x = (x_1, x_2) \in R_+^2$ and show that $F_p(x) \le F_{p^*}(x)$ for all $p \ge 0$.

By the definition of p^* we have for the homogeneous function g:

$$g(x) = x_1^q g\left(1, \frac{x_2}{x_1}\right)\left(\frac{x_2}{x_1}\right)^\beta\left(\frac{x_2}{x_1}\right)^{-\beta} = h\left(\frac{x_2}{x_1}\right)x_1^{q-\beta}x_2^\beta \le h(p^*)x_1^{q-\beta}x_2^\beta.$$

Let $p \le p^*$. Then Theorem 4 gives:
if $x_2 \ge p^* x_1$ then $F_p(x) = g(x) = F_{p^*}(x)$;
if $px_1 \le x_2 < p^* x_1$ then $F_p(x) = g(x) \le h(p^*)x_1^{q-\beta}x_2^\beta = F_{p^*}(x)$;
and if $x_2 < px_1$ then $F_p(x) = h(p)x_1^{q-\beta}x_2^\beta \le h(p^*)x_1^{q-\beta}x_2^\beta = F_{p^*}(x)$. Thus, $F_p(x) \le F_{p^*}(x)$ when $p \le p^*$. Similar arguments prove inequality $F_p(x) \le F_{p^*}(x)$ for $p > p^*$. Thus, maximum for the problem (7.11) is attained at $p = p^*$. From this and the definition of class $\mathcal{M}(\mathcal{G}_0)$ follows Theorem 5.

In order to prove optimality of stopping time τ^* over all Markov times $\mathcal{M}$ we use "verification theorem", based on variational inequalities method (see Bensoussan and Lions, 1987; Oksendal, 1998). We formulate it below.

Let us denote P^x – a distribution of a process ξ_t (in space of trajectories) starting from the initial point $\xi_0 = x$, $\mathbf{E}^x$ – an expectation with relate to the distribution P^x.

Theorem (Oksendal, 1998). *Suppose, there exists a function $\Phi : R_+^m \to R^1$, satisfying the following conditions:*
1) $\Phi \in C^1(R_+^m)$, $\quad \Phi \in C^2(R_+^m \setminus \partial D)$;
here and further $D{=}\{x{\in}R_+^m : \Phi(x){>}g(x)\}$, and ∂D is a boundary of set D,
2) ∂D is locally the graph of Lipschitz function and

$$\mathbf{E}^x \int_0^\infty \chi_{\partial D}(\xi_t)\, dt = 0 \text{ for all } x \in R_+^m;$$

3) $\Phi(x) \geq g(x)$ for all $x \in R_+^m$;
4) $L\Phi = \rho\Phi$ for $x \in D$;
5) $L\Phi \leq \rho\Phi$ for $x \in R_+^m \setminus \bar{D}$ $\quad$ ($\bar{D}$ is a closure of the set D);
6) $\tau_D{=}\inf\{t \geq 0 : \xi_t \notin D\} < \infty$ a.s. (with respect to P^x) for all $x \in R_+^m$;
7) the family $\{g(\xi_\tau)e^{-\rho\tau},\ \tau \leq \tau_D\}$ is uniformly integrable (with respect to P^x) for all $x \in D$.

Then $\tau^* = \tau_D$ is an optimal stopping time for the problem (7.2) over all Markov times, and $\Phi(x)$ is an optimal value of functional.

The proof of Theorem 6. As a candidate we try the function $\Phi(x_1, x_2)$, defined in(7.12). For $x = (x_1, x_2) \in R_+^2,\ x_1 \neq 0$ let denote $p(x) = x_2/x_1$.

Since $h(p^*) > h(p)$ for all $p \neq p^*$, then for $x_2 < p^* x_1$ we have

$$\begin{aligned} \Phi(x_1, x_2) &= h(p^*)x_1^{q-\beta}x_2^\beta > h(p)x_1^q \left(x_2/x_1\right)^\beta \\ &= x_1^q g\left(1, x_2/x_1\right)\left(x_2/x_1\right)^{-\beta}\left(x_2/x_1\right)^\beta = g(x_1, x_2) \end{aligned}$$

(the latter equality follows from the homogeneity of the function g).

Therefore, $\Phi(x){\geq}g(x)$ for all $x{\in}R_+^2$, and the domain $D{=}\{x{\in}R_+^2 : \Phi(x){>}g(x)\}$ coincides with $\{x_2{<}p^*x_1\}{=}\{(x_1, x_2) : 0{\leq}p(x){<}p^*\}$. Furthermore, $\tau_D{=}\inf\{t{\geq}0 : \xi_t{\notin}D\}{=}\inf\{t{\geq}0 : \xi_t^2{\geq}p^*\xi_t^1\}{<}\infty$ a.s. for all $x{\in}R_+^2$ due to Lemma.

Show that condition 7) of the verification theorem follows from (7.9). Indeed, if $\tau \leq \tau_D$ then $\xi_\tau^2 \leq p^*\xi_\tau^1$ and, therefore,

$$\Phi(\xi_\tau)e^{-\rho\tau}{=}h(p^*)(\xi_\tau^1)^q\left(\frac{\xi_\tau^2}{\xi_\tau^1}\right)^\beta e^{-\rho\tau}{\leq}h(p^*)(p^*)^\beta(\xi_\tau^1)^q e^{-\rho\tau}{=}g(1, p^*)(\xi_\tau^1)^q e^{-\rho\tau}.$$

Hence, from an explicit formula for geometric Brownian motion we have:

$$\mathbf{E}^x[\Phi(\xi_\tau)e^{-\rho\tau}]^k \leq g^k(1,p^*)x_1^{kq}\mathbf{E}^x\exp\{[-\rho\tau + q(\alpha_1 - \frac{1}{2}\sigma_1^2)\tau + q\sigma_1 w_\tau^1]k\}$$
$$= g^k(1,p^*)x_1^{kq}\mathbf{E}^x\exp\{-[\rho-\bar{\alpha}_1 q-\frac{1}{2}q^2\sigma_1^2(k-1)]k\tau+kq\sigma_1 w_\tau^1$$
$$-\frac{1}{2}k^2q^2\sigma_1^2\tau\} \leq g^k(1,p^*)x_1^{kq}\mathbf{E}^x\exp\{kq\sigma_1 w_\tau^1 - \frac{1}{2}k^2q^2\sigma_1^2\tau\},$$

if $k > 1$ is chosen such that $\rho - \bar{\alpha}_1 q - \frac{1}{2}q^2\sigma_1^2(k-1) \geq 0$ (due to condition (7.9)). Since $M_t = \exp\{q\sigma_1 w_t^1 - \frac{1}{2}q^2\sigma_1^2 t\}$ is a martingale (see, for example, Karatzas and Shreve, 1991), then $\mathbf{E}M_\tau = \mathbf{E}M_0 = 1$. Thus

$$\sup_{\tau\leq\tau_G}\mathbf{E}^x[\Phi(\xi_\tau)e^{-\rho\tau}]^k \leq g^k(1,p^*)x_1^{kq},$$

and uniform integrability of the family $\{\Phi(\xi_\tau)e^{-\rho\tau},\ \tau\leq\tau_G\}$ holds.

Condition 4) of the verification theorem follows immediately from the definition of function Φ when $x_2 < p^* x_1$.

Let us take now $x_2 \geq p^* x_1$, i.e. $p \geq p^*$ and, therefore, $\Phi(x_1,x_2) = g(x_1,x_2) = x_1^q g(1,p)$. Repeating arguments, similar to those in derivation of equality (7.17), we have:

$$Lg-\rho g = x_1^q\left[\frac{1}{2}p^2 g''_{x_2x_2}(1,p)\tilde{\sigma}^2 + p g'_{x_2}(1,p)(\bar{\alpha}_2-\bar{\alpha}_1-\frac{q-1}{2}\tilde{\sigma}^2) - g(1,p)(\rho-\bar{\alpha}_1 q)\right].$$

Condition (7.13) implies $pg'_{x_2}(1,p) \leq \beta g(1,p)$. From this and (7.9) it follows:

$$x_1^{-q}(Lg-\rho g) \leq \frac{1}{2}p^2 g''_{x_2x_2}(1,p)\tilde{\sigma}^2 + p g'_{x_2}(1,p)\left[\bar{\alpha}_2-\bar{\alpha}_1-\frac{q-1}{2}\tilde{\sigma}^2-\frac{1}{\beta}(\rho-\bar{\alpha}_1 q)\right]$$
$$= \frac{1}{2}p\tilde{\sigma}^2[p g''_{x_2x_2}(1,p) - (\beta-1)g'_{x_2}(1,p)] \leq 0$$

(here we use the fact that β is a root of equation (7.10) and condition (7.14)). Thus, condition 5) is also fulfilled. Hence, all the conditions of the verification theorem hold and, therefore, $\tau_D = \tau^*$ is an optimal stopping time in (7.2) over the class of all Markov moments $\mathcal{M}$.

Concerning the proof of Corollary from Theorem 6 we note that requirement $\rho > \alpha_2$ implies $\beta > 1$.

Theorem 1 follows immediately from Corollary to Theorem 4 and formula (4.7) for discounted firm's profits.

The proof of Theorem 2. Statements 1)–3) follow from the formulas for discounted firm's profits and tax payments into budgets (from the Section 5.2) and Theorem 4 (for the function $g(x) = x_2 - x_1$).

In order to prove 5) put $g(x) = 1$. Then by Theorem 1 we have:

$$\mathbf{E}e^{-\phi\tau^*} = {\pi_0}^{\tilde{\beta}}(p^* I_0)^{-\tilde{\beta}}, \tag{7.19}$$

where $\tilde{\beta}$ is a positive root of the equation

$$0.5\tilde{\sigma}^2\beta(\beta - 1) + (\alpha_2 - \alpha_1 + \sigma_1^2 - r\sigma_1\sigma_2)\beta - \phi = 0. \tag{7.20}$$

Differentiating the equality (7.19) in ϕ, we have

$$\mathbf{E}\tau^* e^{-\phi\tau^*} = {\pi_0}^{\tilde{\beta}}(p^* I_0)^{-\tilde{\beta}}\tilde{\beta}'_\phi \log(p^* I_0/\pi_0). \tag{7.21}$$

Taking a derivative in ϕ at equation (7.20), we obtain

$$\tilde{\beta}'_\rho = \left(\tilde{\sigma}^2\tilde{\beta} + \alpha_2 - \alpha_1 + \sigma_1^2 - r\sigma_1\sigma_2 - 0.5\tilde{\sigma}^2\right)^{-1} = \left(\tilde{\sigma}^2\tilde{\beta} + \alpha_2 - \alpha_1 + 0.5\sigma_1^2 - 0.5\sigma_2^2\right)^{-1}.$$

Put it in (7.21):

$$\mathbf{E}\tau^* e^{-\phi\tau^*} = {\pi_0}^{\tilde{\beta}}(p^* I_0)^{-\tilde{\beta}} \left(\tilde{\sigma}^2\tilde{\beta} + \alpha_2 - \alpha_1 + 0.5\sigma_1^2 - 0.5\sigma_2^2\right)^{-1} \log\left(p^* I_0/\pi_0\right). \tag{7.22}$$

One can easy see from (7.20) that $\tilde{\beta} \to 0$ when $\phi \to 0$. Thus, taking a limit in (7.22) when $\phi \to 0$, we obtain the statement 5). Theorem is proved.

8. CONCLUSIONS

A model of investor's behavior in the real sector of the Russian economy taking into account risk and uncertainty was constructed in this paper. The model considers such elements of the Russian tax system as corporate profit tax, value added tax, social tax, property tax, depreciation mechanisms and tax holidays.

Within the framework of the model an optimal timing rule for investment under NPV criterion was obtained. One of the results was an explicit (analytical) derivation of this rule based on the properties of the tax system (listed above), parameters of the investment project, discount rate and risk process. There was also derived the explicit dependence of the investor's expected net present value and the expected present tax payments from the new firm into the federal and regional budgets on the tax system parameters.

The mutual effect of tax holidays and accelerated depreciation on the investor behavior was also studied. One of the results obtained in the paper were the conditions for which the mutual usage of those exemptions negatively affects the investment activity (thus postponing the investment in the project).

The problem of finding a depreciation policy which maximizes expected tax revenues in regional budget was considered, and the optimal policy was obtained in an explicit form. Optimal depreciation rates were derived for both linear and non-linear methods. The problem of finding an optimal depreciation rate under the restriction on positivity of expected gross profits was studied as well.

Calculations on real adjusted data have shown how the optimal depreciation rates for linear and non-linear methods depend on the expected rate of change of value added, the share of active assets and on the discount rate.

The influence of optimal (from the regional point of view) depreciation policy on the tax payments into the federal budget and on the investor's NPV was studied. This influence was estimated on the basis of the ratio of efficiency indicators under the optimal depreciation rates to the corresponding indicators under some other "reference" depreciation rate. We classify investment projects by groups, within each group indicators of efficiency behave nearly the same way. In particular, it has been shown that the optimal depreciation policy for the region can bring considerable gains both to federal budget and to the investor. (The latter is true if the project has a high share of active assets, an average level of labor intensity in terms of wage per unit of value added, and a less than too high volatility. For those types of projects the optimal depreciation stimulates the investor and forces him to invest earlier).

Within the framework of the proposed model, the authors compared the efficiency of former and new profit taxation schemes for the newly created enterprises. The optimal investment threshold which characterizes the moment of the arrival of the investor, expected tax payments into the consolidated budget, investor's NPV, and expected present tax burden for the enterprise given the optimal investor behavior were all considered as criteria for comparison. The calculations of these indicators using real adjusted data have shown that, for the investment projects with a high share of active assets, the new system of profit taxation substantially outperforms the old one. The difference between the two systems decreases, however, with the increase of projects volatility.

A modified model of investor behavior was proposed for the case of the replacement of property tax by real estate tax (the Novgorod and Tver experiment modeling). The explicit formula for the threshold of investment as a function of all parameters of the model was obtained in the modified model. It was shown that the replacement of property tax by real estate tax would stimulate the arrival of the investor if the share of active asset for the project exceeds a certain critical value (the project is sufficiently capital-intensive). The explicit formula for this threshold was derived.

The possibilities of compensating the NVP risk by the means of a reduction in the profit tax rate and changes in depreciation policy were studied in the framework of investor's model. The authors determined the zone of risk for which losses could not be compensated by any of tax mechanisms listed above.

The analysis of the usage of non-linear depreciation method according to the article 259 of TC RF was conducted. It was shown that the 20%-critical level of the ratio of residual cost to initial cost for switching between non-linear and linear methods, established by the law, did not provide an equivalent choice of depreciation method. It was demonstrated that for the 30%-switching point the equivalence of the depreciation schemes permitted by law can be provided by the relevant recalculation coefficients derived in this report.

Two schemes of losses carry forward induced by taxation were studied. One of them, actual, is established by the new tax code, and the other, a simplified one, was taken from the model. In the deterministic case of a model approximation estimates for replacement of actual scheme by the "simplified" one were obtained.

The construction and investigation of the model of investor behavior gave an impulse to develop a new mathematical approach to the optimal stopping problem for the multi-dimensional diffusion processes. We propose a new variational method for finding an optimal "continuation region", based on the representation of the functional of multi-dimensional diffusion process at the boundary point as a solution of Dirichlet problem. Using this method, the complete solution of the optimal stopping problem for two-dimensional geometric Brownian motion and a homogeneous (of any non-negative degree) objective function was obtained. This permits a detailed analysis of the investor model.

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