КОНСОРЦИУМ ЭКОНОМИЧЕСКИХ ИССЛЕДОВАНИЙ И ОБРАЗОВАНИЯ - РОССИЯ И СНГ
ECONOMICS EDUCATION AND RESEARCH CONSORTIUM – RUSSIA AND CIS

Strong Equilibrium Implementation for a Principal With Heterogeneous Agents*

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EERC project No 00-103

*The author expresses his deepest gratitude to Victor Polterovich, Isabelle Brocas and Shlomo Weber for the continuing interest, ideas, discussions and fruitful critique. The author is happy to thank Alexander Tonis, Alexander Shen, Konstantin Sonin, Gregory Kosenok, Serguei Kokovin and Akhmed Akhmedov for their comments and suggestions.

1. Description of the problem

For principal with many agents, the space of possible contracts is much reacher than just the Cartesian product of individual (peace-rate) contracts spaces. That is, the principal may find it profitable to induce the agents to play a game by choosing their effort levels, not just to run several diversed individual contracts. We will call such contracts by *correlated incentive schemes.*

Exploiting correlated schemes, however, raise questions of implementability. That is, the actual outcome of the second-stage game following a given contract selected (and announced) by the principal may not coinside with the one predicted. Hence, there is a quest for stable and unambigous solution concepts to be relied upon when the secon-stage game is being played. One should take account of cooperation forces among the agents; Nash equilibrium concept is rather too weak to rely upon, needless to say that there may be several Nash equilibria.

Holmstrom (1982) was probably the first who studied team production and correlated incentive schemes. Since and so far, the existing literature considered essentially only one type of correlated incentive contracts, namely, rank-order contracts, or tournaments. These contracts are based solely upon the comparison of individual outputs (the only thing that matters for a tournament is the agent's rank, or his place in the list of outputs). Mookherjee (1984) compared rank-order contracts with individual peace-rate ones, and found that the former ones are welfare enhancing, comparing to the latters. Malcomson (1986) provide additional motivation for analysing the rank order contracts.

However, there is a quest for studying more sophisticated incentive schemes, not just tournaments or individual peace-rate contracts. There are two reasons for that. The first is obvious: the space of all the incentive schemes for the principal with many agents is much reacher than the joint of the two types of contracts mentioned above. Hence, one may suggest that the set of Pareto optimal contracts lies outside both classes just discussed (here, Pareto optimality means that the effort level of no agent could be increased, without decreasing the effort levels of some other agents).

The second reason was already mentioned above, and deals with the implementation issue: rank-order contracts inplement too weak solution concept in the second-stage game. Typically, it is just Nash equilibrium, and which looks especially pessimistic, sometimes there are several of them (see the literature cited above).

One could have assumed, following Mookherjee (1984), that the principal could enforce a given equilibrium allocation (through the focal point effect), but we proceed in another way. Namely, we will search for contracts that implement a solution concept which is prone to coalition formations, and essentially unique. Typically, such contracts will *not* dominate rank-order ones, moreover, they often will be Pareto inferior to the latters, but their inplementation will be unambigous. Then, we will discuss (second-best) optimality of such contracts.

One would suggest that contracts which implement coalitionary stable solutions may look too combersome to have any practical value. It turns out, however, that these contracts were already introduced and studied, although within a different framework. Namely, there is a vast literature on cooperative production and cost sharing mechanisms that pose questions of implementation from a descriptive, positive viewpoint. That is, alternative mechanisms are compared with respect to the quality of solution concept they implement, regardless of any further consideration of efficiency properties of such mechanisms. The basic idea of this research is that the more stable and unambiguous solution concept is being implemented by a given mechanism, the better the mechanism is. Various concepts were proposed, such as core allocations (Bagnoli and Lipman (1989), coalition-proof Nash equilibria and Pareto-optimal Nash equilibria, dominance solvable equilibria and, finally, strong equilibria. The latter, which is the most desirable, is being implemented by the so-called serial cost sharing rule introduced by Littlechild, Owen (1973), and studied extensively in Moulin and Shenker (1992).

We adopt the apparatus of cost sharing theory to the principal-agent relationships. In order to concentrate on implementation issues, we get rid of the hidden information and moral hazard considerations, instead assuming costly implementation of incentive schemes[1].

Proceeding in this way, we first characterize the space of all contracts, or incentive schemes, for the principal with n agents. Then, we introduce and study a certain class of implementation schemes called *multistep strategies* which resemble serial cost sharing rules used in Moulin and Shenker (1992). Precisely, we adopt serial cost sharing of excludable public good from Moulin (1994). Our schemes differ from Moulin's in that there is no explicit cost

[1] Such an approach can be justified by the fact that, in many real situations, the problem faced by the principal is rather not a problem of observability, but that of verification. For instance, it is the case in the tax evasion practice, see Vasin, Panova (2000).

function lying at the base of them. This cost function is prescribed by the contract, and is essentially discontinuous. In spite of these differences, schemes we analyse still admit the unique strong equilibrium; however, discontinuity (hence, nonconvexity) of the underlying cost function makes the apparatus used in Moulin (1994) inapplicable directly to our situation.

In analysing equilibria, we use techniques of monotone comparative statics analysis as in Milgrom and Shannon (1994). It is based on the theory of complete lattices (see Topkis (1998), and Milgrom and Roberts (1990)). One of the by-products of the analysis is a theorem on existence and uniqueness of a strong equilibrium for games of a certain class (which includes second-stage games following any multistep strategy implementation). This class is intimately related to the class of supermodular games (see Milgrom and Roberts (1990)), although one cannot state the inclusion in either direction. We conclude our analysis by characterizing the Pareto optimal incentive schemes for principal with heterogeneous agents, and reducing the problem faced by the principal to the finite-dimentional maximization.

The paper proceeds as follows. Section 2 introduces the basic framework for studying the many-agency problem, which is an n-inspection problem. Section 3 introduces the class of multistep strategies, and discusses their relation to serial cost sharing rules. Section 4 specifies a broad class of games that will be demonstrated to have the unique strong equilibrium. Then, we prove that multistep incentive schemes introduced in Section 3 induce the second-stage game of this class. Section 5 begins with a discussion of various solution concepts, and then states and proves the main theorem of existence and (generally) uniqueness of a strong equilibrium for that class of games; moreover, it is demonstrated that this equilibrium is at the same time the unique Pareto optimal Nash equilibrium, and the unique coalition-proof Nash equilibrium. Section 6 applies the general analysis to a normative question of Pareto optimality properties of implementation schemes (from the principal's viewpoint). Section 7 concludes.

2. n-inspection problem

A basic framework of our analysis is what we call the *n-inspection problem*. There is a principal who hires several (n) agents for some task. Agents could cheat to some degree $z \in [0, 1]$, and the fact of cheating (and its degree, as well) will be revealed by the principal. However, one and only one agent could ultimately be checked and punished, due to lack of time, or to the complexity of the punishment procedure; penalty is assumed to be linear in the cheating parameter. One could treat punishment as a subtraction of the corresponding share of a salary. The question is whether it is possible to reduce cheating by designing (and committing to) a wise *punishment scheme* which attains a probability distribution of being checked to every possible profile of cheating parameters.

It is by no means the only possible model specification allowing to analyse incentive schemes with many agents. For instance, we could have designed an incentive scheme in terms of wages, instead of costs (i.e. penalties), as it is done in e.g. Malcomson (1984). But it is instructive to operate with costs, for on this way we will easily refer to serial cost sharing rules used in Moulin (1994), as well as other public finance literature. In our model, it is assumed that wages are fixed and sufficient to satisfy the participation constraint even at the no-cheating point. The approach we have chosen is applicable to a number of real-world situations, like tax evasion, corruption deterrence etc.

Returning to our basic model, let us assume that the game is two-staged: at the first stage, the principal announces and commits himself to a certain incentive scheme, or contract specifying checking probabilities, conditional upon the profile of cheating parameters realized (hence, observed). Essentially this contract is a mapping from the set $[0, 1]^n$ of all the possible cheating profiles to the set Δ_n of all the probability distributions over the n points (i.e. agents):

$$\lambda : \quad [0, 1]^n \to \Delta_n. \tag{1}$$

The set of all such mappings reflects the principal's strategic opportunities.

At the second stage, agents simultaneously choose their parameters of cheating $z_i \in [0, 1]$. Then, the payoffs are realized, and the principal faces the profile $q = (z_1, \ldots, z_n) \in [0, 1]^n$ (negatively) reflecting the efficiency of the task fulfilled by the agents.

In order to complete the second-stage game description one should specify the agents' preferences. We will assume the agents to be risk neutral, and that their net benefits from cheating are characterized by benefits functions $b_i(z)$ measured in the penalty units. We assume that these functions are continuous, increasing, and satisfy the following property:

$$\forall j > i, \; b_j(z) - b_i(z) \text{ is a nondecreasing function.} \tag{2}$$

This means that if one agent likes leasure more than another one, then the more leasure is available, the greater this difference in pleasure will be. This holds if, for example, $b_i(z)$ are differentiable and $\forall z\ b'_j(z) > b'_i(z)$, that is, the marginal utility from leasure is also greater for the agent who appreciates leasure more. One can notice that the agents are lineary ordered with respect to the propensity to cheating, and this ordering coincides with the natural ordering on the set n for which $1 < 2 < \cdots < n$.

Now, denoting by

$$\lambda(z_1, \ldots, z_n) = (\lambda_1, \ldots, \lambda_n) \in \Delta_n \tag{3}$$

the resulting probability distribution, we can conclude that the payoff to i-th agent equals

$$u_i(q) = u_i(z_1, \ldots, z_n) = b_i(z_i) - z_i \cdot \lambda_i. \tag{4}$$

Recall that λ_i depends on q, hence, on z_i as well.

The principal possesses a certain objective functional $X(q)$, and tries to maximize it by selecting out the best incentive scheme λ. We postpone the discussion of the maximization problem faced by the principal until Section 6, where we will apply the general theory to make normative statements. Our next step is to analyse the second-stage game. In the following section we introduce the subspace of multistep strategies, or incentive schemes, in the set of all the contracts available to the principal.

3. Multistep strategies

There are at least four reasons for subtracting the set of all possible incentive schemes. First of all, one can notice that the strategy set of the principal is enormously large. Indeed, it contains *all* the mappings from $[0, 1]^n$ to Δ_n, essentially including discontinuous ones (Pareto optimal strategies, as a rule, will be discontinuous). Maximization of the principal's objective functional over such a set seems to be intracktable. Secondly, a principal's strategy should not look too complicated, in order to be apprehended by the agents (recall that it is announced to the agents before they choose the degrees of cheating).

Third and forth rationales for subtracting the strategy set were discussed in introduction: they deal with solution concepts to be explored, and questions of uniqueness of equilibrium. Uniqueness is required because if proposed solution concept admits several equilibria, the principal will typically have a headache of guessing which of them will actually be realized. And, of course, it is usually better to have a solution concept as strong as possible, not just a concept of Nash equilibrium, for coalitional forces will probably destroy it[2].

We now introduce a class of strategies which we call *multistep strategies*. All of them induce the second-stage game to have the strong Nash equilibrium, and in general, this equilibrium will be unique. A typical multistep strategy consists of several threshold levels

$$0 \leq \bar{z}_1 < \cdots < \bar{z}_k < 1, \tag{5}$$

[2]If agents are diversed from each other, and cooperation looks impossible, one could implement Nash equilibrium solution. In this case, there is no problem at all within our framework of symmetric information since, as could be easily checked, the first-ranking contract implements the unique Nash equilibrium of no-cheating (this contract consists in deterministic checking of the agent with the maximum cheating parameter, with the equal-probablility gamble between all of them if they are numerous).

and probability distribution

$$(A_1, \dots, A_k) \in \Delta_k \tag{6}$$

over the set of these thresholds (this is *not* λ's, as will be explained below: λ's lie in Δ_n). The better way to specify the multistep strategy corresponding to these data is to tell a fairy tale.

Assume that during the day, the agents are separated to n rooms, and could not observe what others do until the end of the day (in order to exclude dynamics from the story). If a given agent chooses $z \in [0, 1]$, he simply sleeps first z percent of time, wakes up by alarm and works all the rest day. According to the multistep strategy, the principal runs a gamble with corresponding probabilities $\{A_l, l = 1, \dots, k\}$, and in case of l-th outcome, he enters all the rooms right after the moment $\bar{z}_l$ of time. Then, he randomly penalizes one of the agents who is still sleeping, and precisely according to his choice, z (known to the principal, say, from alarm clock at the agent's table).

This story has nothing to do with the real situation, but it helps to become aquainted with the effect of a multistep strategy's implementation. Those who are familiar with Moulin and Shenker (1992) could find that incentive schemes we propose resemble the serial cost sharing principle. This becomes clear from their explanation of serial cost sharing in terms of a "turning lights" story. Strategic properties of serial cost sharing also are close to ours, though not completely coinside with the formers. Namely, our multistep strategies generate multiple Nash equilibria, hence, the predicted outcome of the second-stage game will not be dominance solvable[3].

[3] I ought the idea that multistep strategies are essentially cost sharing rules to Shlomo

The most closely connected to ours are serial cost sharing rules for public goods provision in Moulin (1994). To explain why, let us interpret checking probabilities assigned to agents as costs of cheating. Then, we have the following symbolic picture: agents should bear together costs of producing the maximal cheating parameter from the collection $q = (z_1, \ldots, z_n)$. Costs are stepwise: it costs A_l to produce any additional amount $\delta z \in (0, \bar{z}_{l+1} - \bar{z}_l]$ above $\bar{z}_l$ (where we set $\bar{z}_{n+1} := 1$). Costs are then divided exactly as in Moulin (1994). The specification of the cost function is a strategic choice of the principal. The only problem is nonconvexity and discontinuity of any cost function constructed on this way.

Now, we could easily formalize multistep strategies, directly adopting formulas from Moulin (1994). Probabilities λ_i depend on the parameters of a multistep strategy in the following way:

$$\lambda_i = \sum_{\{l:\bar{z}_l < z_i\}} \frac{A_l}{\#\{j : \bar{z}_l < z_j\}}. \tag{7}$$

That is, agents with $z > \bar{z}_l$ divide together costs A_l for increasing the cheating degree above the threshold z_l. Together with the formula (4), this completes the description of the second-stage game implemented by a typical multistep strategy. Next section introduces a class of games which postulates axiomatically the properties shared by payoff functions of any such game.

Weber.

4. Supermodular iterative monotone games

Consider a game Γ of n players. The set of all the players is denoted by $N = \{1, 2, \ldots, n\}$. Let a strategy set of all players be one and the same compact subset $Z \in \mathbf{R}$ (for instance, a closed segment, like $[0, 1]$). Denote by $q \in Z^n$ the profile of strategies chosen by the agents, $q = (z_1, \ldots, z_n)$, and by $q_{-i} \in Z^{n-1}$ the profile of strategies chosen by agents other than i. The set Z^k inherits an ordering from the set Z (which looses its linearity for $k \geq 2$), and forms a complete lattice, with respect to this partial ordering.

Besides, we will need to preserve the linear ordering of the set N of the players (or agents, synomically). It gives us a right to say that a profile $q = (z_1, \ldots, z_n)$ is *monotone* if $z_1 \leq \cdots \leq z_n$. The set $\Re$ of monotone profiles also forms a complete lattice (trivial exercise). Also, we denote by q_- a monotone profile of $(n-1)$ strategies.

The definition given below encorporates the main constructions and ideas of monotone comparative statics theory (see Milgrom and Shannon (1994), Topkis (1998), and Milgrom and Roberts (1990)), with those from Moulin and Shenker (1992) inspired by the idea of defining serial cost sharing rules axiomatically.

Definition. The game described below is called a *Supermodular Iterative Monotone game* (or, simply, a SIM-game) iff the payoff functions $u_i(z_1, \ldots, z_n) = u_i(z_i; q_{-i})$ satisfy the following five properties.

1. **Anonimuity.** $\forall i \quad u(z_i; q_{-i})$ depends only on the collection

$$q_- = \{z_j | j \neq i\} \in Z^{n-1}, \tag{8}$$

that is, does not change under permutations of the other agents. (In other words: it does not matter for Lesha whether Masha chooses z and Ira chooses w, or vise-versa.)

This condition is adopted from Moulin and Shenker (1992) where it was used for a characterization of serial cost sharing. We therefore could write $u(z_i; q_-)$ instead of $u(z_i; q_{-i})$, where q_- denotes a monotone transformation of q_{-i}. Moreover, it will be useful to rewrite payoff functions in the form of only one function $u(z; [i, q_-])$. In what follows, "$>$" means "$\succeq$, but not $=$".

2. **Weak Single crossing property (WSCP) in** $(z; [i, q_-])$. Exactly as in Milgrom and Shannon (1994): for $z' > z$ and $(j, q'_-) \succeq (i, q_-)$, if

$$u(z'; [i, q_-]) \geq u(z; [i, q_-]) \tag{9}$$

then

$$u(z'; [j, q'_-]) \geq u(z; [j, q'_-]). \tag{10}$$

This property (precisely, its *strong* form) is crucial for well-behaving comparative statics, as shown in Milgrom, Shannon (1994); we will need its relaxed analogue for monotonic properties of agents' *choices*.

3. **Monotonicity.** $\forall z \in Z$ and $(j, q'_-) \succeq (i, q_-)$ we have

$$u(z; [j, q'_-]) \geq u(z; [i, q_-]). \tag{11}$$

This resembles the "complementarity" of choices made by different agents (compare with Milgrom and Roberts (1990)), and, at the same time, the fact that agents with high indices occupate better positions than those with low ones. This assumption will guarantee monotonic properties of the agents' *payoffs*.

4. **Ordinal semi-dependence, or Iterativity.** Together with the first property, this one is being taken from Moulin and Shenker (1992): a payoff u_i of i-th agent does not depend on the choices not lower than his own choice. Formally, let us denote by q_-^z the profile q_- in which all the components higher than z are replaced by z. Then, we require that $\forall i$

$$u_i(z, q_-) = u_i(z, q_-^z). \tag{12}$$

   In Moulin and Shenker (1992), this assumption was placed on cost sharing rules; we switch it directly to the payoff functions.

   Note that continuity is not required, and in fact, payoff functions arose from the n-inspection problem will be discontinuous. We therefore need a regularity condition which will guarantee that individual maximization problems nonempty and nicely-behaved solution sets. Such a condition is well-known for this sort of models, and essentially states that the function evaluated on the limit of a monotone sequence is not lower than the limit of its valuations on the terms of this sequence.

5. **Regularity condition.** Payoff function $u(z; [i, q_-])$, being considered as a function of $z \in Z$, is an order upper semi-continuous (see precise formulation in Milgrom and Roberts (1990)).

   This regularity condition is fulfilled, for example, if payoff functions are continuous except for a finite number of jumps *down*, and are left-side continuous in the points of jumps (a mathematical triviality). It turns out that all these properties are satisfied for the second-stage game of the n-inspection problem with a multistep strategy implemented by the principal. This property completes the definition of a SIM-game.

Theorem 1 *For any multistep strategy (5), (6), the second-stage game of the n-inspection problem is a SIM-game with* $Z = [0, 1]$.

Proof is given in Appendix, together with the graphical illustration and a discussion of an agent's maximization problem.

5. Existence and uniqueness of a strong equilibrium for SIM-games

Various solution concepts purifying Nash equilibrium

There are many solution concepts strengthening the notion of Nash equilibrium. The commong problem most of them share is the generic nonexistence. As for Nash equilibrium, it typically exists, at least in convex situations, because there are "the same number of" equations determining them and variables to be determined. As for strengthening solutions, they exist only under serious assumptions on the game form.

At the same time, if the *family of games* is under consideration, and if this family is reach enough, there is a hope that some games within the family have nonempty sets of solutions that are stronger than Nash equilibrium. Our n-inspection story is an example: in the huge set of all the possible incentive schemes for principal with many agents, there is a subset of contracts that implement coalitionary proof solutions, namely, those resulting in SIM-games at the second stage. Before analysing these SIM-games, let us discuss various strengthenings (or *purifications* of the notion of Nash equilibrium. We begin with the strongest solution concept.

Strong equilibrium (SE). This notion was first introduced by Aumann (1959). A given allocation is said to form a strong equilibrium if there are no coalitions of agents that could simultaneously change the strategies of its members so as to increase the payoffs for all of them, given that other agents hold on using strategies of the initial allocation. If no coalition could increase the payoff of even one of its members holding others at the initial level of utility, we call such an allocation *superstrong equilibrium* (SSE). This is probably the most stable and unambiguous solution concept for one-shot games. If the principal implements the scheme characterized by the only SE in the second-stage game, he can safely predict the outcome of this game to coincide with this SE.

Pareto optimal Nash Equilibrium (PONE). This simply claims that, gethering altogether, agents could not do better for all of them then in the Nash equilibrium allocation. If they could not increase the utility of even one of them holding others at the initial level of utility, this is the *strong* PONE, or simply SPONE. Every SE is at the same time PONE, and SSE is SPONE. In PONE (and SPONE), the only working coalitions is the vast majority plus individuals; in Nash equilibrium, the only working coalitions are individuals. In the celebrated Prizonner's Dilemma, there are no PONE, neither SPONE (hence, neither SSE and SE): the only (dominance-solvable) Nash equilibrium is not Pareto optimal.

Coalition-proof Nash equilibrium (CPNE). This notion is close to the SE (or SSE), and differs in that it allows only those coalition formations that are self-sustainable: coalition proposing better outcomes to their participants should not generate stimula for second-round group deviations, again provided the latters are sustainable. The formal definition bears the inductive nature, and could be found in pioneering work by Bernheim, Whinston and Peleg (1987). Notice that, defining CPNE, we require that coalitions offer strict increases in utility to *all* their participants. Every SE is a CPNE, whereas there is no inclusion between CPNE and PONE sets, in general.

A disussion of various solution concepts of cooperative game theory also could be found in Ichiishi (1993). One probably finds it difficult to cope with all these notions and their interrelations. Fortunately, there is no need in such an inquirry now: it turns out that every SIM-game have a (generally, unique) strong equilibrium (SE) which, in addition, could be obtained iteratively by a very simple, straightforward procedure.

SIM-games: existence of a strong equilibrium

We begin our analysis of SIM-games by the following statement.

Lemma 1 *In every SIM-game, the agent's payoff function* $u(z;[i,q_-])$ *has a nonempty set* $f[i,q_-]$ *of argmaxima, for all pairs* $(i,q_-) \in N \times Z^{n-1}$. *This set* $f[i,q_-]$ *forms a complete sublattice in* Z. *As a consequence, there always exists a maximum in this set, which we call* $z[i,q_-]$: $\forall (i,q_-) \in N \times Z^{n-1}$

$$\begin{aligned} &\forall z \in Z \quad u(z[i,q_-];[i,q_-]) \geq u(z;[i,q_-]), \textit{and} \\ &\forall z > z[i,q_-] \quad u(z[i,q_-];[i,q_-]) > u(z;[i,q_-]). \end{aligned} \tag{13}$$

Proof could be found in Milgrom and Roberts (1990).

Next, we conduct the comparative statics with respect to $[i,q_-]$, that is, analyse the responce functions $z[i,q_-]$ introduced above.

Lemma 2 *The function $z[i, q_-]$ is nondecreasing with respect to both arguments (or equivalently, being considered as a function over the partially ordered set $N \times Z^{n-1}$), and so is the induced payoff function $u(z[i, q_-]; [i, q_-])$.*

Proof is a trivial consequence of the monotonicity and WSCP properties. For instance, to prove monotonicity of $z[i, q_-]$, one needs to replace $z' = z[i, q_-]$ in WSCP, which gives us that for $[j, q'_-] \geq [i, q_-]$, $z[i, q_-]$ is at least as good as $z < z[i, q_-]$, hence, $z[j, q'_-] \geq z[i, q_-]$ (recall the definition of a function $z[i, q_-]$ as the maximum of argmaxima set $f[i, q_-]$). The same for utility, using standard envelope argument.

Various modifications of this obvious assertion could be found in Topkis (1978), who is probably the first to state this property explicitely. One corollary of this result is very important.

Corollary. The mapping $\bar{f} : Z^n \to Z^n$, defined by

$$\bar{f}(q) := (z[1, q_{-1}], \ldots, z[n, q_{-n}]), \tag{14}$$

is a monotone transformation of a complete lattice Z^n.

Now we apply the celebrated Birkgoff —Knaster —Tarski theorem to the mapping $\bar{f} : Z^n \to Z^n$.

Theorem 2 *For any complete lattice H and its monotone transformation h, the set of fixed points is nonempty, and forms a complete sublattice in H. The SUP of the sublattice of equilibria is a transfinite limit of a sequence $SUP(H) \to h(SUP(H)) \to h^2(SUP(H)) \to \ldots$.*

Proof could be found in Tarski (1955), or in Topkis (1998).

This gives us the existence theorem for Nash equilibria, since obviously every fixed point of the mapping $\bar{f}$ is a Nash equilibrium. The reverse is not true, because $\bar{f}$ is *not* a best-responce correspondence but just a *maximal best responce* function. We introduce the whole best responce correspondence by

$$f(q) \;:=\; f[1, q_1] \times \cdots \times f[n, q_n] \subset Z^n \tag{15}$$

where $f[i, q_-]$ is the argmaximum set of $u(z; [i, q_-])$, with respect to z. Nash equilibria are precisely fixpoints of this correspondence. Denote by q^* the SUP of the lattice of all the fixpoints of the mapping $\bar{f}$. The following lemma asserts that q^* is the SUP of the set of all Nash equilibria.

Lemma 3 *For every Nash equilibrium $\tilde{q} \in Z^n$ we have $q^* \succeq \tilde{q}$.*

Proof. If $\tilde{q}$ is an equilibrium, it means that $\tilde{q} \in f(q)$, with $\bar{f}(q)$ obviously being the SUP of the set $f(q)$. Hence, we have that $\bar{f}(\tilde{q}) \preceq \tilde{q}$. Applying transfinite times $\bar{f}$ to the obvious inequality $I \succeq \tilde{q}$ and taking transfinite limits (see Topkis (1998)), we obtain the required property (recall that translim$(\bar{f}^t(I))$ coinsides with q^*).

Turning to the next item, we state the basic proposition, which proof is complicated and postponed until the Appendix.

Lemma 4 *The maximal equilibrium $q^* = (z_1^*, \ldots, z_n^*)$ could be obtained through the following iterative inductive procedure (in the first string, [1;...] means that this is the best responce of the agent 1 to the profile $q_- = (S, \ldots, S) \in Z^{N-1}$, where $S = SUP(Z)$; other strings being treated by the* *analogy. We have $I = (S, \dots, S)$):*

$$\begin{aligned} z_1^* &= z[1; S, \dots, S]; \\ z_2^* &= z[2; z_1^*, S, \dots, S]; \\ &\dots\dots \\ z_k^* &= z[k; z_1^*, \dots, z_{k-1}^*, S, \dots, S]; \\ &\dots\dots \\ z_N^* &= z[N; z_1^*, \dots, z_{N-1}^*]. \end{aligned} \tag{16}$$

Notice that it is not the iterative process $I \to \bar{f}(I) \to \bar{f}(\bar{f}(I)) \dots$. For the calculational purposes, the process in (16) is much more convenient. It turns out that its theoretical value is great as well: it is a powerful technical tool. Armed with this tool, let us turn to the next item.

Lemma 5 *The maximal equilibrium q^* is at least as good* for all the agents *as any allocation $q = (z_1, \dots, z_N)$ such that $q \preceq q^*$, hence, as any other equilibrium allocation.*

Proof. Indeed, we have the following chain of inequalities $\forall i$:

$$u_i(z_i; q_{-i}) \le u_i(z_i; q_{-i}^*) \le u_i(z_i^*; q_i^*), \tag{17}$$

of which the former one is stated in lemma 2, while the latter expresses the equilibrium nature of q^*.

Next theorem is the basic result of the paper.

Theorem 3 *The maximal equilibrium q^* is a superstrong equilibrium of any SIM-game.*

Proof. To prove that q^* is an SSE, assume that there is a coalition $J \subset N$ of deviators, and values $\{z'_i | \, i \in J\}$ of strategies for its members such that, $\forall i \in J$,

$$u_i(z'_i; q'_{-i}) \geq u_i(q^*), \tag{18}$$

with at least one strict inequality (by q' we denote the allocation $(\{z'_i | \, i \in J\}; \{z^*_i | \, i \notin J\}))$. Take the first agent $i \in J$ who increases his choice: $z'_i > z^*_i$, and $z'_j \leq z^*_j \quad \forall j < i$ (provided such agents ever exist). For this agent one has the following contradiction:

$$\begin{aligned} u_i(z'_i; q'_{-i}) \leq u_i(z'_i; z^*_1, \dots, z^*_{i-1}, S, \dots, S) < \\ < u_i(z^*_i; z^*_1, \dots, z^*_{i-1}, S, \dots, S) = u_i(q^*). \end{aligned} \tag{19}$$

Let us comment on this. First inequality follows from the fact that the profile from Z^{n-1} in the second term is by-componently not lower than the profile q'_{-i}, by the definition of i; second stems from the fact that z^*_i is a *maximal* best responce to $(z^*_1, \dots, z^*_{i-1}, S, \dots, S)$, by the construction of lemma 4. The last is the ordinal semi-dependence property.

It means that there are no agents whose strategy $z'_i > z^*_i$. But in this case, we have $q' \preceq q^*$, and so, q^* is at least as good as q' for all the agents, hence, $\forall i \in J$ as well. So, there are no possibilities to collude and block the allocation q^*. The proof of theorem 2 is complete.

Corollary. The sets of SE, PONE, SPONE and CPNE are all non-empty and contain a maximal element (which is q^*).

Indeed, any SSE belongs to every set mentioned above, so all of them contain q^*. At the other hand, all these sets itself are subsets of the set NE of all the Nash equilibria, of which q^* is a supremum.

Questions of uniqueness

Let us now turn to the question of uniqueness of equilibrium. To start with, we state the following proposition which is a direct sequence of monotonicity and WSCP properties.

Lemma 6 *In the SSE q^*, the first agent attains absolute (i.e. first-best) maximum of utility. Inductively, every agent i attains his second-best maximum of utility, constrained by the fact that first $(i-1)$ agents attain their second-best in a similar fascion (with the first one attaining his first-best).*

This lemma leaves us without a hope that q^* could be a unique PONE: every allocation $(z_1^*, \ldots, z_i^*, z_i^*, \ldots, z_i^*)$ is a PONE (generally) because the attempt to increase individually the degree z over common z_i^*-value results in a very high probability of checking, hence, it is unprofitable; so, such an allocation typically is a Nash equilibrium. And this equilibrium is Pareto optimal since the first i agents attain their constrained maximal utility levels, as in the allocation q^*.

However, such an allocation generally is not a SPONE: switching together to q^* does not decrease utilities of all the agents, and, provided some regularity conditions hold, increase the utilities of the agents $i+1, \ldots, n$. These conditions are summarized in the following *theorem of uniqueness*.

Theorem 4 *Assume that all the sets*

$$
\begin{aligned}
&f[1; S, \dots, S];\\
&f[2; z_1^*, S, \dots, S];\\
&\dots\dots\\
&f[k; z_1^*, \dots, z_{k-1}^*, S, \dots, S];\\
&\dots\dots\\
&f[N; z_1^*, \dots, z_{N-1}^*]
\end{aligned}
\tag{20}
$$

are single-valued (the so-called First Regularity Condition*, or FRC); and that the function* $u(z; [i, q_-])$ *is strictly increasing in those components of* q_- *that are lower than* z *(this is* Second Regularity Condition*, or SRC). Then, all the sets SSE, SE, SPONE and CPNE are single-valued.*

Proof is straitforward: Given an allocation $\tilde{q}$ which is a candidate for SSE, SE, SPONE or CPNE, we find the first agent i whose $\tilde{z}_i < z_i^*$ (recall that $\tilde{q} \preceq q^*$, since at least $\tilde{q}$ pretends to be a Nash equilibrium). Then, we form a coalition of $(i+1, \dots, n)$ and offer them their equilibrium strategies z_j^*. In this case, it is easy to check that FRC and SRC imply the strict increase in the utility level of all the coalition members. For CPNE, we need to establish additionally that the new allocation is sustainable. But the strategies for deviators are SSE-allocation strategies, hence, form a CPNE. Proof is over.

6. Optimal multistep strategies

Let us now turn back to the N-inspection problem, and analyse strategic opportunities of the principal. First of all, let us discuss the uniqueness properties for this application. It turns that FRC is typically satisfied for the n-inspection problem.

Lemma 7 *Except for a subset of measure* 0*, all the multistep strategies induce a second-stage game satisfying FRC.*

Proof is purely mathematical: we notice that every condition on the set $f[i;q_-]$ to be single-valued is a (nongenerate) equation on the data (5), (6). A standard calculus concludes the proof.

As for SRC, generally it *does not* hold for the n-inspection problem. The obstacle is a stepwise character of the utility functions. Nevertheless, it could be replaced by the requirement that every set $f[i,q_-]$ contains no more than one point within every interval $(\bar{z}_l, \bar{z}_{l+1}]$. It turns out that, under quite natural conditions, this is satisfied with probability one. Below is presented a result whose proof is analogously purely mathematical and is skipped.

Theorem 5 *If every benefit function* $b_i(z)$ *is iether convex or concave on* $[0,1]$*, then, except for null-set, all the correspondences SE, SSE, SPONE, and CPNE are single-valued (being defined over the set of all multistep strategies).*

Now we are going to analyse the optimal multistep strategies. No matter which objective functional $X(q)$ is being used, a question arises concerning the Pareto-efficiency of a given strategy: whether it is the least costly (and the easiest) way to implement a given profile, q. The next (and the last) theorem answers to this question.

Theorem 6 *For every SSE profile q^* of a multistep strategy (5), (6) the following N-stepped strategy $(z_1^*, \dots, z_N^*; \bar{A}_1, \dots, \bar{A}_N)$ implements a profile $\tilde{q}^* \preceq q^*$ (possibly, coinciding with the latter):*

$$\bar{A}_i = \sum_{\bar{z}_l \in [z_i^*, z_{i+1}^*)} A_l, \tag{21}$$

Proof consists of applying the iterative procedure of lemma 4 step by step, and involving inductive argument. See Savvateev (2003).

Corollary. In characterizing Pareto optimal multistep strategies for principal with N agents, it is sufficient to consider the $(2N-1)$-dimensional manifold (to be more precise, a manifold with corners) of N-stepped strategies.

This corollary allows to reformulate the problem of choosing the optimal strategy of the principal as the finite-dimentional maximization, whatever functional $X(q)$ is being used. Even without further research, one can design at least a computer program. Namely, we take a sufficiently frequent net on the $(2N-1)$-dimentional cube, and for every vertex of this net the iterative process is conducted which results in the strong Nash equilibrium corresponding to that vertex (i.e. to the following strategy of the principal). According to Theorem 3, this process requires maximum N iterations. Once equilibrium is approached, the objective functional is applied to it, and then the best strategy is being selected.

Special cases $N = 1$ and $N = 2$ are solved in Savvateev (2003) explicitely. Also, one can find there a complete characterization of implementable allocations q^*, and a system of equations characterizing the multistep strategy implementing a given allocation q^*, if it is implementable. However, to perform such an analysis, we need an additional assumption that benefits functions are linear in the cheating parameter: $b_i(z) = b_i \cdot z$, where $b_1 \leq \cdots \leq b_N$. It is assumed, further on, that the principal observes the bundle $(b_1, \ldots, b_N)$, but not individual characteristics b_i for every agent.

Interesting aspects arise on this way concerning the so-called *chain-reaction effect* which is responsible for the implementability of a given allocation. This effect has various policy implications (see Savvateev (2003)).

7. Conclusion

In the paper, a principal-agent model with many agents is analysed which is free of informational assymetries. The primary focus was made on the nature of a strategic space of the principal, especially on its useful subspace of the so-called *multistep incentive schemes.* These schemes generally induce the unique strong equilibrium allocation in the game that is played by the agents when they choose their effort levels.

Also, the conditions guaranteeing the existence and the uniqueness of the strong equilibrium, as well as of some alternative solution concepts, are generalized and presented in a purely game-theoretic form. The kind of games under study are in a close connection with a supermodular games studied in the literature on comparative statics. Additional assumptions made came from the serial cost sharing method used in a public finance literature. It is the combination of these two approaches that results in the existence and uniqueness of a strong equilibrium.

After the general game-theoretic analysis, the principal-agent framework with many agents is reconsidered, and Pareto optimal incentive schemes are characterized. It turns out that, despite the huge and intractable nature of the space of all the incentive schemes, the set of Pareto optimal multistep contracts is a finite-dimentional and easily observable submanifold.

Also, iterative procedure is described for obtaining the strong equilibrium for any multistep strategy. This procedure both helps a lot in theoretical investigation of the games under study, and provides a way to solve for the optimal incentive scheme numerically, given a specific form of an objective functional of the principal. Possible applications of the theory are given.

Appendix

Proof of theorem 1. For convenience, we reproduce here formulas specifying the sort of games under study. Payoffs are given by

$$u_i(q) \;=\; u_i(z_1, \ldots, z_N) = b_i(z_i) - z_i \cdot \lambda_i(z_1, \ldots, z_N), \tag{22}$$

where λ_i is being determined by the parameters of a multistep strategy through the following formula

$$\lambda_i = \sum_{\{l: \bar{z}_l < z_i\}} \frac{A_l}{\#\{j : \bar{z}_l < z_j\}}. \tag{23}$$

We need to prove that the game specified by (22) and (23) satisfy 5 properties of section 4. Let us start with the regularity condition. Namely, we will show that payoff functions are continuus elsewhere except for points z_l, $l = 1, 2, \ldots, k$ where it experiences a jump *down*, and is left-side continuous at these points. As was mentioned in Section 4, this is sufficient for order upper semi-continuity.

The proof is a graphical one. The figure 1a below illustrates the components of the payoff function of i-th agent as a function of his choice variable, z (for the case of a two-stepped incentive scheme $[(z_1, z_2); (A_1, A_2)]$). The continuous line starting at zero-point represents his benefits function, $b_i(z)$. The cost function, which is $z \cdot \lambda(z; q_-)$, is peacewise linear but discontinuous, and has jumps *up* exactly at threshold levels z_1 and z_2. The resulting payoff function is demonstrated at the figure 1b, and it is apparent that it satisfies the required property.

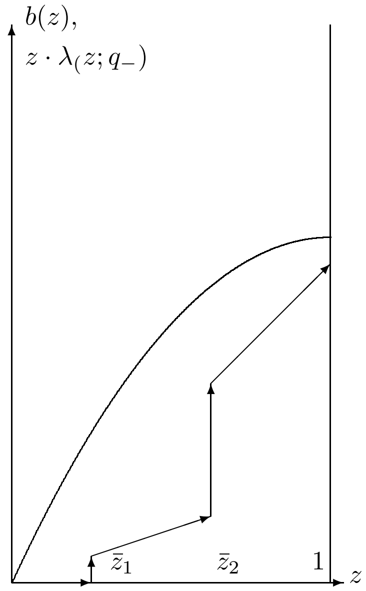

Figure 1a. Components of a payoff function

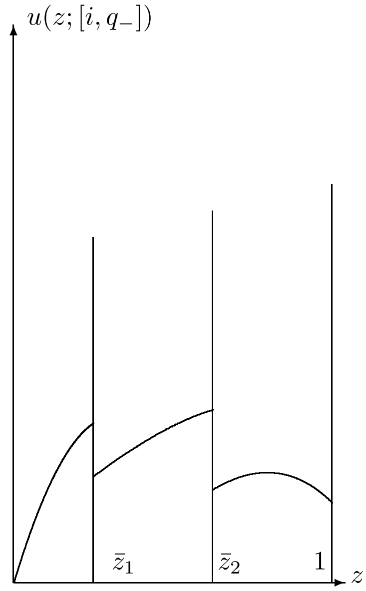

Figure 1b. A typical payoff function

Let us now turn to the first four properties. As for the anonimuity, it follows from formulas (22), (23) immediately, since the only thing that matters is the *numbers* of agents choosing cheating degrees within the corresponding intervals.

The same with the ordinal semi-dependence (or iterativity). Every term in (23) with $\bar{z}_l < z_i$ will not be altered if those who choose $z \geq z_i$ will switch to $z = z_i$ instead, whereas other terms equal zero.

Let us prove the monotonicity of the payoff functions. If one changes agent i to $j > i$ holding q_- unaltered, the balance equals $b_j(z) - b_i(z)$ and is nonnegative, by the definition. If, alternvely, we switch from q_- to $q'_- \succeq q_-$, this makes *every* term in (23) nonincreasing, for the numerator holds the same while the denominator could only increase (given $\bar{z}_l$, the number of the agents whose choices exceed $\bar{z}_l$ increases or remains the same). Therefore, λ_i nonincreases, which implies that

$u_i(z; q_-)$ nondecreases.

As for the WSCP, consider $z' > z$. If we have $u(z'; [i, q_-]) \geq u(z; [i, q_-])$, it means that

$$b_i(z') - b_i(z) + z\lambda_i(z; q_-) - z'\lambda_i(z'; q_-) \geq 0. \tag{24}$$

Actually λ_i does depend only on its argument, and not on i, thus decomposing the proof into two distinct parts: increasing i to $j > i$, and increased q_- to $q'_- \succeq q_-$. Increasing i is a simple case: by the assumption, the function $b_j(z) - b_i(z)$ is nondecreasing in z, so we have

$$b_j(z') - b_i(z') \geq b_j(z) - b_i(z) \Leftrightarrow b_j(z') - b_j(z) \geq b_i(z') - b_i(z). \tag{25}$$

Increasing q_- requires a little more work. Our goal again is to establish that (omitting index i, due to already proved anonimuity)

$$z\lambda(z; q'_-) - z'\lambda(z'; q'_-) \geq z\lambda(z; q_-) - z'\lambda(z'; q_-). \tag{26}$$

Rearranging, one can see that this is equivalent to

$$z'(\lambda(z'; q_-) - \lambda(z'; q'_-)) \geq z(\lambda(z; q_-) - \lambda(z; q'_-). \tag{27}$$

As $z' > z$, it is sufficient to demonstrate that expressions in brackets are compared to the needed direction. Both present a sum of several members of a type

$$\frac{A_l}{\#\{j : \bar{z}_l < z_j\}} - \frac{A_l}{\#\{j : \bar{z}_l < z'_j\}}, \tag{28}$$

and notice that all these terms are *nonnegative*, due to the argument used to prove monotonicity. Moreover, these terms are identical in both sides of (27)! The only difference consists in their numericity: the LHS of (27) contains not less of them, since $z' > z$. So, increasing q_- also results in increase of the term in (24) it responds for. Summing up, we conclude that (24) nondecreases when switching from $[i, q_-]$ to $[j, q'_-] \geq [i, q_-]$. Hence, it is still nonnegative. WSCP property is established.

The proof of theorem 1 is complete.

Proof of lemma 4. We first prove that the allocation $(z_1^*, \dots, z_n^*)$ defined by the iterative procedure (16) is a monotone profile. We proceed inductively: assume that the sub-profile $(z_1^*, \dots, z_i^*)$ is monotone, and consider z_{i+1}^*. We have

$$\begin{gathered} z_{i+1}^* = z[i+1; z_1^*, \dots, z_i^*, S, \dots, S] \geq \\ z[i; z_1^*, \dots, z_i^*, S, \dots, S] = z[i; z_1^*, \dots, z_{i-1}^*, S, \dots, S] = z_i^*, \end{gathered} \tag{29}$$

where the last equality follows easily from ordinal semi-dependence axiom, in a way similar to that used in proving theorem 3.

Now, we establish that the allocation $(z_1^*, \dots, z_n^*)$ is a Nash equilibrium. Again, we use the same argument: inductively, and using monotonicity, we have that

$$\begin{gathered} z_i^* = z[i; z_1^*, \dots, z_i^*, S, \dots, S] = z[i; z_1^*, \dots, z_{i-1}^*, z_i^*, \dots, z_i^*] = \\ z[i; z_1^*, \dots, z_{i-1}^*, z_{i+1}^*, \dots, z_n^*]. \end{gathered} \tag{30}$$

And lastly, we will prove (again, inductively) that the allocation $q^* \succeq \tilde{q}$ for every Nash equilibrium allocation $\tilde{q}$. Namely, if it is proved that $z_j^* \geq \tilde{z}_j$ for $j = 1, \dots, i-1$, then

$$\begin{gathered} z_i^* = z[i; z_1^*, \dots, z_{i-1}^*, S, \dots, S] \geq \\ z[i; \tilde{z}_1, \dots, \tilde{z}_{i-1}, \tilde{z}_{i+1}, \dots, \tilde{z}_n] \geq \tilde{z}_i, \end{gathered} \tag{31}$$

because $z[i; q_{-i}] = max\{f[i; \tilde{q}_{-i}]\}$, and $\tilde{z}_i \in f[i; \tilde{q}_{-i}]$.

Summing up, we have proved that q^*, obtained by the inductive procedure (16), is the maximal Nash equilibrium. Proof of lemma 4 is complete.

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