CONTENTS

# 1. INTRODUCTION: MOTIVATION AND BRIEF DESCRIPTION OF APPROACHES

A tendency towards a decline in investment has been characteristic of the Russian economy since the beginning of the 90s. The volume of the annual Russian investment market is estimated as 20bn dollars. According to the Russian Federation State Committee on Statistics, during 1992-1997 only 21.8bn dollars was invested in non-financial sectors in Russia. Lately, the dynamics of investment in Russia have become positive, but the total volume decreased in 1997 by 5% (in comparison, in 1996 the rate of decline was 18.1%), and in the first quarter of 1998, in comparison with the same period of 1997, it was down by 7% . The dynamics of direct foreign investment in Russia characterise the increase in the credibility of Russia to foreign investors, but their absolute value is still very insignificant (less than 1% of GNP). During 1997, foreign investment in the real sector of the Russian economy was only 26.4 dollars per capita (in 1996 – 14.2 dollars), i.e. dozens of times less than in eastern European countries (Sotsialno-ekonomicheskoe ..., 1997).

The reasons for the unfavourable investment climate in Russia are well-known. They are, first of all, to do with the unstable political and economic situation (changes in the legal system are difficult to forecast, and the actions of the federal executive bodies and the subjects of the federation are not synchronised); the too heavy tax burden; and the criminal situation (including in the enterprise sphere). Russian and foreign investors are afraid that their projects in Russia will not be effective because

- laws that contradict previous agreements can be adopted (for example the elimination of tax credits);
- decisions on the submission of property rights to the investor can be declared illegal;
- the economic policy of the country will change as a result of any change in the political situation;
- after investments are made, new limitations on the entrepreneur may appear which will lead to rejection of the further use of the enterprise.

The main methods of creating incentives for investment activity are also well known. Among them are:

- development of the system of state guarantees to investors (for example through the creation of guarantee and mortgage funds which accumulate the financial resources of the state, firms and individuals);
- development of the investment insurance system;
- tax benefits to the investors.

The first two items have to do mainly with the regulation of investment at state level, which is now in the first stage of development in Russia (see, for example, Petrakov et al. (1997)). Tax policy incentives are widely used in Russia and have already given positive results in some regions. Lately, the mass media and legal bodies have started a campaign against tax benefits (drawing connections, in many cases justifiably, between tax benefits and corruption), but a favourable tax policy is the only realistic and available instrument to attract foreign and domestic entrepreneurs.

Additional arguments in favour of this position are the results of the latest poll of multinationals investing in European countries carried out by the consultancy firm "Deloitte-Touche Tomatzu International" . The purpose of the poll was to reveal the degree of influence of tax incentives on investment decisions.

From the answers obtained from 100 corporations, one can reach the conclusion that tax incentives might not be a decisive factor in the adoption of decisions on allocation of new investments (more important are political stability, the stability of the national currency, and the quality of labour) but, all other things being equal, those countries which provide tax benefits and information on them are preferred. One of the main conclusions of the company experts is that "tax incentives are one of the most important weapons for incentives to foreign investors".

Of the tax incentives, the most important from the point of view of the respondents are low corporate income taxes. These are valued much more highly than accelerated amortisation and export exemptions. It should be noted that corporate income tax is a significant portion of the total tax burden and is one of the most important revenue portions of the budget; its share of total tax revenues is about 1/4 (25.8% in 1995, 20.4% in 1996, and 17.8% in 1997 – according to Goskomstat data). According to the data of the Russian Federation State Tax Service in 1995, corporate income tax revenues were the largest in 10 out of the 11 Russian territories (in only the West

Siberian region were they slightly less than the revenues from value added tax). Lately there has appeared a trend towards a decrease in the share of corporate income tax; but revenues from this source in the federal budget still increased by 13% during the first quarter of 1998 in comparison with the same period of the previous year (at the same time, revenues from value added tax decreased by 21%, and excise duties by 22%) .

The current Russian corporate income tax system is characterised by federal and regional (territorial) tax rates as well as different tax exemptions. The federal tax rate is 13%, while the regional rate is determined by local authorities, but should not be greater than 22%. In some regions, the local tax rate is lower than the upper limit, for example in St.Petersburg (20%), Tatarstan (19%), Nizhni Novgorod region (21%), Perm' region (17.5%). A description of the main tax exemptions in the Russian regions is given in Appendix A.

Among the exemptions which decrease the effective tax rate, the most popular are tax holidays, i.e. full or partial remission from income tax within certain intervals following investment. These exemptions have been used in south-eastern Asia, eastern Europe and some western European countries. One of the most successful was the tax holiday programme in Puerto Rico, which was initiated in 1949 (Bond (1981)). In the survey by Egorova et al. (1997), it is pointed out that 10-15 year tax holidays have been introduced for so-called pioneer corporations specialising in high-tech production in Singapore. In Italy, beginning in 1986, there have been tax holidays for firms that have been established in Southern Italy, where such exemptions are a regional incentive.

In Russia, tax holidays are used in more than two dozen regions (including the Novgorod region, Chuvashia, Tatarstan, Tver' region, Samara, Kaliningrad, Yekaterinburg, etc.). Mainly, they have been influenced by the growth in the economic and political independence of the subjects of federation, which have created new possibilities for the attraction of investors (including foreign ones) for concrete projects, with the help of the adoption of regional laws on tax and other exemptions, the creation of local guarantee funds, easier bureaucratic procedures, etc.

Unlike the early 90s, during the most recent two years foreign investors have begun to pay attention to regional investment projects.

The most vivid example is the Novgorod region, which has created a real investment boom . In December 1994, the regional Duma adopted a law

"On tax exemptions for enterprises located on the territory of the Novgorod region". According to this law, enterprises with foreign investments which are engaged in production and are registered on the territory of the region are exempt from all regional taxes during the payback period. The law gives regional enterprises exemptions of, on average, up to 30% of total tax contributions to the local budget. Starting from January 1, 1997, four districts of the region have been declared tax-free zones and the federal part of income tax out of the local budget is redeemed to them. As a result, more than 160 enterprises have been registered in the region with foreign investment (mostly from Germany, Finland, and the United States). The Western-Russian Regional Venture Fund of the European Bank for Reconstruction and Development gave 30m dollars for the financing of investment projects, and 20m dollars were subsidised by the Italian government. Consequently, during the three years between 1994 and 1996, the volume of foreign investment increased from 3.5m dollars to 154m dollars. Russian companies invested in the economy of the region 40bn rubles. According to the expert evaluation of the World Bank, the Novgorod region is among the six most attractive Russian regions for foreign investments.

In the economic literature, there are many publications on tax incentives for firms already in existence. To be precise, they deal with the investigation of the influence of tax rates and other incentives, such as accelerated depreciation, tax allowances on re-investment in production, and tax credits (see, for example, Kueschnigg (1989), Daly et al. (1993), Feltenstein and Shah (1995)). When capital is invested abroad and foreign subsidiaries of multinationals are created, there appears an additional problem with tax credits and deferrals (depending on the relationship between home and host tax systems) with the purpose of avoiding double taxation (see Hines (1994), Mintz and Tsiopoulos (1994) and the corresponding references there).

Unlike the above-mentioned topics, our paper deals with tax incentives for investment in new firms. In order to clarify the motivation for our research, we should briefly explain its contents.

Let us consider an investment project that assumes the creation of a new enterprise in a certain region that produces particular goods and consumes particular resources. We will limit our investigation to the case when investments are direct and irreversible, i.e. they can not be withdrawn and used for other purposes. This investment project can be imagined as a certain consequence of expenditures and outputs in physical units (the technological description of the project) in time.

Considering prices for input and output production, the investor can calculate expected profits before actually making an investment (virtual profits). When calculating net present value (NPV) the investor should consider the various factors of risk and uncertainty based on the dynamics of virtual profits. First of all, the prices and demands of production can fluctuate stochastically depending on the situation in the market (the "market or economic" risk). After investment, the profits of firms already in existence are negatively influenced by different negative events, such as default, unpredictable actions of the authorities, the lack of developed infrastructure, crime, etc. The abovementioned factors are called "institutional risks". Also, investor behaviour depends on an evaluation of the probability of the loss of the firm (and investments also) as a result of any change in the political course and the creation of unacceptable situations. This factor is called "political risk".

The behaviour of the investor is presumed to be rational in the sense that, while looking at the virtual profit from a given project (and evaluating the situation in a certain region), a decision can be adopted either to invest or to postpone for some time in order to receive information on the situation in the economic environment (for example on the change in virtual profits). So, the objective of the investor is to choose the optimal moment for investment depending on the information that has been obtained before this time.

The region can actively influence investor behaviour, accelerating presence with the help of tax exemptions, for example appropriate intervals of tax holidays. We consider that the purpose of the region for a given project is the maximisation of the discounted tax payments into the regional budget during the existence of the enterprise.

Within the proposed framework of interaction between the region and the investor, the following tasks are investigated:

- determination of the optimal investment rule as a function of all the parameters of the problem;
- investigation of the dependence of the main economic indicators of the region, investor and federal centre on the parameters of uncertainty, risk, and tax exemptions (comparative statics);

- comparison of different factors (the parameters of the project, federal and regional rates of income tax, tax exemptions, discount values and political risk) according to their influence on investment activities;
- evaluation of the effectiveness of the mechanism of tax exemptions, depending on the above-mentioned factors;
- determination of the range of parameters (that characterise these factors) inside which tax exemptions can compensate for unfavourable risk factors and provide positive effects for the regional and federal budgets;
- comparison of the "optimal" tax exemptions (i.e. those within the proposed model) with those existing in reality in the different regions of Russia.

The starting point for our research is the McDonald-Siegel model (McDonald and Siegel (1986)) which was developed and described in detail in Dixit and Pindyck (1994). These papers deal with the model of investor behaviour in which the profit after investment in a certain project is described by a stochastic process (geometric Brownian motion), while investments are considered to be irreversible. The purpose is to find the optimal moment for investment.

The approach to this problem proposed in these papers is related to Contingent Claims Analysis (CCA). Investment is interpreted as the purchase of American call options on the right to make investments in the future. The expiry date of this option is the optimal moment of investment. One of the main assumptions of such a model concerns the availability on the securities market of a financial asset, the price of which is completely correlated with the market price of the realised investment project. It is assumed that the financial market is in equilibrium, and in particular that it satisfies the conditions of the known Capital Assets Pricing Model (CAPM).

The model of the investor proposed in this paper is an extension of the McDonald-Siegel model by the inclusion of the existing Russian corporate income tax system, as well as political and institutional risks.

However, the approach to the investigation of this problem, related to the use of CCA methods and the CAPM, is not suitable for economies with undeveloped financial markets (including the Russian economy). For this reason, we use other methods, based on the optimal stopping theory for stochastic processes, for the investigation of the investor model. In this case, the optimal rule (according to the NPV criterion) is interpreted as the moment of optimal stopping of the process of observed virtual profits. Thus, the lack of investment (investment waiting) is the consequence of the assumptions of the rational behaviour of the investor. For the solution of the optimal stopping problem (for establishing the optimal investment level), instead of the traditional heuristic "smooth pasting" method (see, for example, McDonald and Siegel (1986), Dixit and Pindyck (1994)), we propose a rigorous approach based on the direct evaluation and further variation of an optimised function. The investment rule, obtained in analytical form, allows the region to compare the different principles of the determination of tax holidays, in particular to optimise their duration from the point of view of tax payments into the regional budget, depending on the parameters of the tax system.

This paper has the following structure. In Section 2 we describe a model of an investor who is interested in a certain investment project and waiting for the appropriate moment to invest. Investigation of this model is carried out in Section 4, under certain assumptions about the process of obtaining profit from the project that are made in Section 3. It turned out that the solution of the problem of the proposed investor can be found in an explicit form. We use obtained analytical formulas both for the analysis of the dependencies of the economic indicators (related to the project) on the parameters of the model (which are characterised by uncertainty, risk, and tax exemptions) as well as for the comparison of the influence of different factors (the parameters of the model) on investment activity. In Section 5, different principles of the determination of tax holidays are analysed. "Optimal tax holidays" that maximise discounted tax payments into the regional budget are defined. On the basis of calculations (using adjusted real data), a mechanism of optimal tax holidays is compared with existing principles in Russian regions. In Section 6 we make some conclusions about the proposed model. Finally, Appendix A contains a list of the main tax exemptions (for investors) in Russian regions and in Appendix B we give proofs of the main mathematical results (Theorems 1 and 2 from Section 4.1).

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## 2. THE BASIC MODEL – THE INVESTOR'S PROBLEM

Under consideration as an object of investment will be a project for the creation of a new enterprise (involved in production) in a certain region of a country.[1] Investments $I$, necessary for the project, are considered to be instantaneous and irreversible, so that they can not be withdrawn from the project and used for other purposes after the project was started (sunk costs).

One can think of an investment project as a certain sequence of costs and outputs expressed in units (the technological description of the project). For this reason, while looking at current prices on both resources and output, the investor can evaluate the profit from the project; this would be profit after the investment was made and, before the moment of investment, one can call it profit, i.e. the hypothetical profit under the conditions that the investment would be made at the initial (zero) moment.[2]

The most important feature of the model is the assumption that at any moment the investor can either the project and start with the investment, or the decision before obtaining new information on its environment (prices of the product, demand, etc.), i.e. on the changes in the virtual profit from the project. For example, if someone wishes to invest in the creation of a plant for fuel production, the prices of which increase, it makes sense to delay the investment in order to receive greater virtual profit (but not for too long because of the time discount effects).

The economic environment can be influenced by different stochastic factors (uncertainty in market prices, demand, etc.). For this reason, we consider that the profit from the project (including all taxes and payments except corporate income tax)[3] can be described by a stochastic process $\Pi = (\pi_t,\ 0 \leq t < \infty)$. As usual it is supposed that the process $\Pi$ is defined in some probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and that it is measurable regarding the flow of $\sigma$-algebras $(\mathcal{F}_t, t \geq 0)$, where $\mathcal{F}_t$ can be considered as information on the system up to the moment $t$.

As for the lifetime of the project (the duration of the existence of the new firm), this is considered infinite, although in an unstable socio-political environment the investor is afraid that the project may cease to bring any profit after a certain time following the investment as a result of changes in the political or economic course of the country, and the situation will be such that investor will have to refuse further use of the enterprise. This factor is naturally called , and in our model it is taken into account in that the investor will receive revenues only within a certain period of time after making the investment, while the duration of this interval $L$ () is a random variable.

The tax system influences investor behaviour significantly. We restrict our investigation to corporate income taxes that bring about one-quarter of all the tax revenues of the state budget. According to current Russian laws, it is characterised by state (federal) and territorial (regional) tax rates as well as a set of tax benefits. As can be seen from Appendix A, the majority of exemptions are tax holidays, i.e. full or partial[4] exemption from regional corporate income tax for some time after the firm is created and profit obtained. In this paper, we will focus on tax holidays, so the tax system can be represented as a triplet $(\gamma_f, \gamma_r, \nu)$, where $\gamma_f$ and $\gamma_r$ are, respectively, the federal and regional corporate income tax rates and $\nu$ is the duration of the tax holidays.

Suppose that investment in the project is started at moment $\tau$.

The present value of investor returns from the project can be written by the following formula:

$$\begin{aligned} V_\tau &= \mathbf{E}\left( \int\limits_{\tau}^{\tau+\min(\nu,L)} (1-\gamma_f)\pi_t e^{-\rho(t-\tau)}\, dt \right. \\ &+ \left. \int\limits_{\tau+\nu}^{\tau+\max(\nu,L)} (1-\gamma_f-\gamma_r)\pi_t e^{-\rho(t-\tau)}\, dt \,\middle|\, \mathcal{F}_\tau \right), \end{aligned} \tag{1}$$

where $\rho$ is the investor discount rate, $L$ is the lifetime of the revenues for the investor, and the notation $\mathbf{E}(\cdot|\mathcal{F}_\tau)$ stands for the conditional expectation provided by information about the system until moment $\tau$.

The purpose of the investor is to find a moment for investment (the rule of investing) which depends on previous (but not future) observations of the environment, so that its net present value (NPV) will be maximal within the given tax system, i.e:

$$\mathbf{E}\,(V_\tau - I)\,e^{-\rho\tau} \to \max_{\tau}, \tag{2}$$

where $\mathbf{E} = \mathbf{E}(\cdot|\mathcal{F}_0)$ is the sign of the expectation (provided the known data about the system at moment $t = 0$), and the maximum is considered over all the "investment rules", i.e. $\tau$, depending only on the observations of the environment (among them the virtual profit from the project), up to this point (the Markovian moment, i.e. such that an event $\{\tau \leq t\}$ belongs to $\sigma$-algebra $\mathcal{F}_t$ for all $t$).

At the same time, we can calculate the tax payments into the budget that can be made by the project after investment. These tax revenues depend on the behaviour of the firm after period $L$ when investor income revenues end.

One of the possible assumptions (that we are going to explore further) is that, although after an interval of length $L$ a certain event happens, say expropriation, the firm itself continues to work without any change in performance and to pay taxes. In this case, the expected tax payments from the firm into , discounted to moment $\tau$, are equal to:

$$T_\tau^f = \mathbf{E}\left(\int_\tau^\infty \gamma_f \pi_t e^{-\eta(t-\tau)}\, dt \;\middle|\; \mathcal{F}_\tau\right), \tag{3}$$

and into the –

$$T_\tau^r = \mathbf{E}\left(\int_{\tau+\nu}^\infty \gamma_r \pi_t e^{-\eta(t-\tau)} dt \;\middle|\; \mathcal{F}_\tau\right), \tag{4}$$

where $\eta$ is the budget discount rate (that compares the values of the budget revenues in time), which can, in general, be different from the discount $\rho$.

The other possible assumption is that, after the period $L$, the firm ceases to function and to pay taxes. In this case, the expected tax payments into the federal and regional budgets, discounted to the moment of investment, are described by the formulas:

$$\widetilde{T}_\tau^f = \mathbf{E}\left(\int_\tau^{\tau+L} \gamma_f \pi_t e^{-\eta(t-\tau)}\, dt \;\middle|\; \mathcal{F}_\tau\right), \tag{3'}$$

$$\widetilde{T}_\tau^r = \mathbf{E}\left(\int_{\tau+\nu}^{\tau+\max(\nu,L)} \gamma_r \pi_t e^{-\eta(t-\tau)} dt \;\middle|\; \mathcal{F}_\tau\right). \tag{4'}$$

Of course, there exists the intermediate case (which may be the most realistic one), where expropriation (after period $L$) changes the profits from the firm (e.g. reduces it) without terminating its activity. In this case, the corresponding expected tax payments can be written as:

$$\widetilde{\widetilde{T}}_\tau^f = \mathbf{E}\left( \int\limits_{\tau}^{\tau+L} \gamma_f \pi_t e^{-\eta(t-\tau)}\, dt + \int\limits_{\tau+L}^{\infty} \gamma_f \widetilde{\pi}_t e^{-\eta(t-\tau)}\, dt \,\middle|\, \mathcal{F}_\tau \right), \qquad (3'')$$

$$\widetilde{\widetilde{T}}_\tau^r = \mathbf{E}\left( \int\limits_{\tau+\nu}^{\tau+\max(\nu,L)} \gamma_r \pi_t e^{-\eta(t-\tau)}\, dt + \int\limits_{\tau+\max(\nu,L)}^{\infty} \gamma_r \widehat{\pi}_t e^{-\eta(t-\tau)}\, dt \,\middle|\, \mathcal{F}_\tau \right), \qquad (4'')$$

where $\widetilde{\pi}_t$ is the process of the firm's profits after expropriation.

## 3. MAIN ASSUMPTIONS

As was noted above, $L$ is considered as a random variable. We will assume that it does not depend on the flow of project revenues and that it has an exponential distribution with the parameter $\delta$, i.e. it has a density $p(L) = \delta e^{-\delta L}$. Parameter $\delta$ can be interpreted as the rate of a "political risk", since it characterises the probability of a "catastrophe" (that there are no investor revenues) within a small interval of time, under the condition that it never occurred in the past, i.e. $\mathbf{P}\{t < L < t + dt \,|L > t\} = \delta dt$. Note that if $\delta = 0$, then the lifetime of investor revenues becomes infinite (there is no political risk).

$I$ is constant in time. Such an assumption does not restrict general considerations, and, for example, the case of exponential growth of investment (in time) can be reduced to the "constant case" with a simple shift of parameters. In McDonald and Siegel (1986), an even more general case was considered where $I$ evolved according to geometric Brownian motion, but this does not lead to a totally new pattern, it only makes the formulas more complicated.

is described by the stochastic process $\Pi = (\pi_t,\ t \geq 0)$. In order to specify it, let us define $R(t, \Delta t) = \dfrac{\pi_{t+\Delta t} - \pi_t}{\pi_t}$ – the rate of profits growth at interval

$(t, t + \Delta t)$. We will consider the process $\Pi$ which satisfies the following assumptions:

**(P1)** $R(t, \Delta t)$ does not depend on $\mathcal{F}_t$ – "the past" of the system until moment $t$;

**(P2)** distribution of $R(t, \Delta t)$ does not depend on moment $t$;

**(P3)** almost all trajectories $\pi_t$ are positive and continuous in $t$.

The conditions described here reflect some "extreme" properties of the environment in which the project exists. For example, (P1) means that the rates of growth of profits can not be predicted for certain on the basis of "the past", and (P2) – that they are regular stochastically. Condition (P3) is the most restrictive and selects projects, which provide positive profits immediately after investment.[5]

These conditions do seem extremely strong, but it turned out that they determine the stochastic process of profits $\Pi$ in a particular way, standard for financial models.

**Proposition 1.** $(\pi_t, t \geq 0)$

$$d\pi_t = \pi_t(\alpha\, dt + \sigma\, dw_t), \tag{5}$$

$$\pi_t = \pi_0 \exp\{(\alpha - \frac{\sigma^2}{2})t + \sigma w_t\}. \tag{6}$$

$\pi_0 \alpha \sigma \sigma \geq 0 w_t$

*Proof.* Define the random process $X_t$ as follows: $X_t = \log \frac{\pi_t}{\pi_0}$, $t \geq 0$. Since $X_{t+\Delta t} - X_t = \log \frac{\pi_{t+\Delta t}}{\pi_t} = \log(1 + R(t, \Delta t))$, then by virtue of the properties **(P1)**–**(P3)** the process $(X_t,\ t \geq 0)$ is a continuous homogeneous process with independent increments and the initial point $X_0 = 0$. Thus, the known results on the representation of continuous random processes (see, for example, Gikhman and Skorokhod (1977), pp.240,34) imply that $X_t$ is a linear function of the Wiener process, i.e. $X_t = at + bw_t$, when $a, b$ are real ($b$ in general is non-negative, because of the symmetry of the Wiener process). Hence we get equation (6), while the equivalence of (5) and (6) follows directly from the Ito formula (see Gikhman and Skorokhod (1977)). Q.E.D.

The parameters of the geometric Brownian motion $\alpha$ and $\sigma$ have a natural economic interpretation, namely:

$\alpha = \lim\limits_{\Delta t \to 0} \dfrac{1}{\Delta t} \mathbf{E}\, R(t, \Delta t)$ is an expected instantaneous rate of profits growth;

$\sigma^2 = \lim\limits_{\Delta t \to 0} \dfrac{1}{\Delta t} \mathbf{D}\, R(t, \Delta t)$ is an instantaneous variance of rate of profits growth (volatility of the project).

We should emphasise that the rate of growth in profits does not have to be positive. Negative $\alpha$ means that the profits flow decreases with time (on average), while nevertheless remaining positive; and when $\alpha = 0$, it changes around its mean $\pi_0$.

The process of geometric Brownian motion was introduced for the first time, probably, by P. Samuelson (Samuelson (1965)), who called it "an economic Brownian motion" and considered it as the most appropriate way of describing the evolution of prices in the economy. The hypothesis of geometric Brownian motion is also the basis of the modern description of securities prices in the financial markets. In particular, it lies in the foundations of the well-known option pricing theory (Black and Scholes (1973)).

Now we can write explicit formulas for the investor's Present Value (1) and for tax payments into the budgets. Using the known formula $\mathbf{E}e^{h\xi} = e^{h^2 D\xi/2}$ for the Gaussian random variable $\xi$ with mean zero and variance $D\xi$, the equation (6) for geometric Brownian motion implies:

$$\mathbf{E}(\pi_t \,|\, \mathcal{F}_\tau) = \pi_\tau e^{(\alpha - \frac{\sigma^2}{2})(t-\tau)} \mathbf{E} \exp\{\sigma(w_t - w_\tau)\} = \pi_\tau e^{\alpha(t-\tau)}, \quad t \geq \tau. \tag{7}$$

Applying the Fubini Theorem, we have:

$$V_\tau \quad = \quad \int\limits_0^\nu \int\limits_\tau^{\tau+L} (1-\gamma_f)\mathbf{E}(\pi_t \,|\, \mathcal{F}_\tau) e^{-\rho(t-\tau)} p(L)\, dt\, dL$$

$$
\begin{aligned}
&+ \int\limits_{\nu}^{\infty} \left( \int\limits_{\tau}^{\tau+\nu} (1-\gamma_f)\mathbf{E}(\pi_t|\,\mathcal{F}_\tau)e^{-\rho(t-\tau)}\,dt \right. \\
&+ \left. \int\limits_{\tau+\nu}^{\tau+L} (1-\gamma_f-\gamma_r)\mathbf{E}(\pi_t|\,\mathcal{F}_\tau)e^{-\rho(t-\tau)}\,dt \right) p(L)\,dL \\
&= \pi_\tau \int\limits_0^{\nu} \int\limits_0^{L} (1-\gamma_f)e^{-(\rho-\alpha)t}\,dt\,p(L)\,dL \\
&+ \pi_\tau \int\limits_{\nu}^{\infty} \left( \int\limits_0^{\nu} (1-\gamma_f)e^{-(\rho-\alpha)t}\,dt + \int\limits_{\nu}^{L} (1-\gamma_f-\gamma_r)e^{-(\rho-\alpha)t}\,dt \right) \\
&\times\ p(L)\,dL = \pi_\tau \left( \int\limits_0^{\nu} \int\limits_t^{\infty} (1-\gamma_f)e^{-(\rho-\alpha)t}p(L)\,dL\,dt \right. \\
&+ \left. \int\limits_{\nu}^{\infty} \int\limits_t^{\infty} (1-\gamma_f-\gamma_r)e^{-(\rho-\alpha)t}p(L)\,dL\,dt \right) \\
&= \pi_\tau \left( \int\limits_0^{\nu} (1-\gamma_f)e^{-(\rho+\delta-\alpha)t}\,dt + \int\limits_{\nu}^{\infty} (1-\gamma_f-\gamma_r)e^{-(\rho+\delta-\alpha)t}\,dt \right) \\
&= \pi_\tau \frac{1-\widehat{\gamma}}{\mu-\alpha}, \qquad \text{where } \ \mu=\rho+\delta, \quad \widehat{\gamma}=\gamma_f+\gamma_r e^{-(\mu-\alpha)\nu}. \qquad (8)
\end{aligned}
$$

The two last lines in these equations show that $\mu = \rho + \delta$ can be viewed as of the investor, and $\delta$ as the "political risk premium" (on the problem of measuring the effect of political risk, see also Clark (1997)).

Analogously, we can obtain explicit formulas for the expected tax payments into the federal and the regional budgets (3) and (4):

$$
\begin{aligned}
T_\tau^f &= \int_\tau^\infty \gamma_f \, e^{-\eta(t-\tau)} \mathbf{E}(\pi_t | \mathcal{F}_\tau) \, dt = \gamma_f \, \pi_\tau \int_\tau^\infty e^{-(\eta-\alpha)(t-\tau)} dt \\
&= \frac{\gamma_f \, \pi_\tau}{\eta - \alpha}, \qquad (9) \\
T_\tau^r &= \int_{\tau+\nu}^\infty \gamma_r \, e^{-\eta(t-\tau)} \mathbf{E}(\pi_t | \mathcal{F}_\tau) \, dt = \gamma_r \, \pi_\tau \int_{\tau+\nu}^\infty e^{-(\eta-\alpha)(t-\tau)} dt \\
&= \frac{\gamma_r \, \pi_\tau}{\eta - \alpha} e^{-(\eta-\alpha)\nu}. \qquad (10)
\end{aligned}
$$

For the other variants of tax payments $(3')$ and $(4')$ one can also easily obtain:

$$
\begin{aligned}
\widetilde{T}_\tau^f &= \int_0^\infty \int_\tau^{\tau+L} \gamma_f \, e^{-\eta(t-\tau)} \mathbf{E}(\pi_t | \mathcal{F}_\tau) p(L) \, dt \, dL \\
&= \gamma_f \, \pi_\tau \int_0^\infty \int_\tau^{\tau+L} e^{-(\eta-\alpha)(t-\tau)} dt p(L) \, dL = \frac{\gamma_f \, \pi_\tau}{\eta - \alpha} \int_0^\infty \left(1 - e^{-(\eta-\alpha)L}\right) \\
&\times \;\; p(L) \, dL = \frac{\gamma_f \, \pi_\tau}{\eta - \alpha} \left(1 - \frac{\delta}{\eta + \delta - \alpha}\right) = \frac{\gamma_f \, \pi_\tau}{\eta + \delta - \alpha}, \\
\widetilde{T}_\tau^r &= \int_\nu^\infty \int_{\tau+\nu}^{\tau+L} \gamma_r \, e^{-\eta(t-\tau)} \mathbf{E}(\pi_t | \mathcal{F}_\tau) p(L) \, dt \, dL = \gamma_r \, \pi_\tau \int_\nu^\infty \int_\nu^L e^{-(\eta-\alpha)t} dt \\
&\times \;\; p(L) \, dL = \frac{\gamma_r \, \pi_\tau}{\eta - \alpha} e^{-(\eta-\alpha)\nu} \int_0^\infty \left(1 - e^{-(\eta-\alpha)(L-\nu)}\right) p(L+\nu) \, dL \\
&= \frac{\gamma_r \, \pi_\tau}{\eta + \delta - \alpha} e^{-(\eta+\delta-\alpha)\nu}.
\end{aligned}
$$

From these relationships, one can see that the case when the firm ceases to function after the investor stops receiving incomes can be reduced (from the point of view of total tax payments into the budgets) to the situation when the firm continues functioning with an adjusted budget discount $\eta' = \eta + \delta$. For this reason, we will consider the first case as the main one.

## 4. INVESTIGATION OF THE BASIC MODEL

In this Section we provide a solution for the model formulated above. As it turned out, it can be obtained in an explicit (analytical) form. On the basis of the obtained formulas, we provide a theoretical analysis of the model.

### 4.1. SOLUTION OF THE INVESTOR PROBLEM

The problem which the investor faces is an optimal stopping problem for the stochastic process. The relevant theory is well developed (see, for example, Shiryaev (1969)), but there are very few problems which have a solution in an explicit form, and problem (2) belongs to this type.

Let $\beta(\theta)$ be a positive root of the quadratic equation:

$$\frac{1}{2}\sigma^2\beta(\beta-1)+\alpha\beta-\theta=0. \tag{11}$$

We should point out that $\beta(\theta)>1$ whenever $\theta>\alpha$. If $\sigma>0$, then, obviously:

$$\beta(\theta)=\frac{1}{2}-\frac{\alpha}{\sigma^2}+\sqrt{\left(\frac{\alpha}{\sigma^2}-\frac{1}{2}\right)^2+\frac{2\theta}{\sigma^2}}.$$

If $\sigma=0$, then $\beta(\theta)=\theta/\alpha$ whenever $\alpha>0$, and there is no positive root of the equation (11) whenever $\alpha\le 0$, but we find it convenient to consider $\beta(\theta)=\infty$.

We will denote $\beta=\beta(\rho)$, $\tilde{\beta}=\beta(\eta)$.

**Theorem 1.** $\rho>\alpha$

$$\tau^*=\min\{t\ge 0:\pi_t\ge\pi^*\}^6 \tag{12}$$

$$\pi^*=kI\frac{\rho+\delta-\alpha}{1-\widehat{\gamma}},k=\frac{\beta}{\beta-1},\widehat{\gamma}=\gamma_f+\gamma_r e^{-(\rho+\delta-\alpha)\nu}$$

(The proof of this theorem, as well as of Theorem 2 below, is contained in Appendix B.)

Theorem 1 shows that the optimal moment for the investment begins when virtual profit achieves a critical level $\pi^*$. Formulas of this type (for the case of geometric Brownian motion) are given in McDonald and Siegel (1986) and Dixit and Pindyck (1994) (for a more simple model of the investor), and probably follow from the results McKean (1965) and Merton (1973) obtained in option pricing theory.

One can look at equality (12) from the investor's point of view (see formula (8)), namely, the optimal moment for the investment $\tau^*$ coincides with the moment when the investor's Present Value $V_\tau$ achieves threshold level $kI$. This means that the classical investment rule – $V_\tau I$ does not work any longer for this model, since the investments can be postponed, and it should be modified in the following way: $k = \dfrac{\beta}{\beta - 1} k > 1$. Detailed analysis of this phenomenon and its connections with the known Jorgenson rules and Tobin's ratio $q$ can be found in Dixit and Pindyck (1994).

In order to avoid the trivial moment of investment $\tau^* = 0$, we will further suppose that the initial value of the profit $\pi_0$ satisfies $\pi_0 < \pi^*$.

If we know the optimal moment for the investment, we can find the expected optimal income for the investor as well as the relevant expected tax payments from the project into the federal and regional budgets. Let us denote the discounted net income of the investor under the condition of optimal behaviour, i.e. the maximum value of the function in (2)), as $\mathfrak{V}$, let $\mathfrak{T}^f = \mathbf{E}\left(T^f_{\tau^*} e^{-\eta\tau^*}\right)$ be the discounted tax payments into the federal budget under optimal behaviour, and $\mathfrak{T}^r$ be the similar value for the tax payments into the regional budget.

We should point out that the optimal moment of investment $\tau^*$ is not always finite, i.e. the project can remain non-invested. Let us define $P^* = \mathbf{P}\{\tau^* < \infty\}$ as the probability of investing in the project (at some finite moment of time).

If the project will be invested in, then $\mathbf{E}\tau^*$ which characterises investment activity (with regard to the project), i.e. the time possibilities of the investor's entry into the system, is also interesting for study.

**Theorem 2.** $\min\{\rho, \eta\} > \alpha$

$\mathfrak{V} = (k-1) I \left(\frac{\pi_0}{\pi^*}\right)^{\beta} \mathfrak{T}^f = \frac{\gamma_f \pi_0}{\eta - \alpha} \left(\frac{\pi_0}{\pi^*}\right)^{\tilde{\beta}-1}$

$$\mathfrak{T}^r = \frac{\gamma_r \pi_0}{\eta - \alpha} \left(\frac{\pi_0}{\pi^*}\right)^{\tilde{\beta}-1} e^{-(\eta-\alpha)\nu}$$

$$P^* = 1\alpha \geq \tfrac{1}{2}\sigma^2 P^* = \left(\frac{\pi_0}{\pi^*}\right)^{1-2\alpha/\sigma^2} \alpha < \tfrac{1}{2}\sigma^2$$

$$\mathbf{E}\tau^* = \frac{1}{\alpha - \sigma^2/2} \log \frac{\pi^*}{\pi_0} \alpha > \tfrac{1}{2}\sigma^2 \mathbf{E}\tau^* = \infty\alpha \leq \tfrac{1}{2}\sigma^2$$

$$\pi^* \tilde{\beta} = \beta(\eta)$$

From that one can see that, in any case (with probability one) only "strong profitable" projects (the parameters of which are connected by the relation $\alpha \geq \frac{1}{2}\sigma^2$) will be invested in. In particular, for the deterministic case ($\sigma = 0$) those will be all the projects with $\alpha \geq 0$. At the same time, the other projects (including all those with negative $\alpha$) will stay, with a positive probability, without investment (at any moment of time).

### 4.2. INFLUENCE OF TAX HOLIDAYS

At this point, we will examine how tax policy can influence investor and tax payments into the federal and regional budgets.

As relative  it is natural to compare the ratio of the optimal investor's NPV (or, tax payments into budgets, respectively) under tax holidays to the corresponding income without tax holidays.

In this Section, we will use the following notation $\tau^*(\nu), \pi^*(\nu)$, $\mathfrak{V}(\nu)$ etc. in order to highlight the dependence of the optimal moment and level of investment, investor's income, and the other indicators mentioned above on the duration of tax holidays $\nu$.

With the help of explicit formulas from Theorem 2, we can obtain the following expressions for such estimates:

$$\mathcal{E}^i = \frac{\mathfrak{V}(\nu)}{\mathfrak{V}(0)} = \zeta^{\beta} \qquad \text{(investor)}, \tag{13}$$

$$\mathcal{E}^f = \frac{\mathfrak{T}^f(\nu)}{\mathfrak{T}^f(0)} = \zeta^{\tilde{\beta}-1} \qquad \text{(federal budget)}, \tag{14}$$

$$\mathcal{E}^r = \frac{\mathfrak{T}^r(\nu)}{\mathfrak{T}^r(0)} = \zeta^{\tilde{\beta}-1} e^{-(\eta-\alpha)\nu} \qquad \text{(regional budget)}, \tag{15}$$

where $\zeta = \dfrac{1-\widehat{\gamma}(\nu)}{1-\widehat{\gamma}(0)} = \dfrac{1-\gamma_f - \gamma_r e^{-(\mu-\alpha)\nu}}{1-\gamma_f-\gamma_r}$, and $\beta$, $\tilde{\beta}$ were defined as at the beginning of Section 4.1. (16)

If $\alpha > \sigma^2/2$ then the expected speed-up of investment in circumstances of tax holidays, in comparison with the case when they are absent, is expressed by the formula:

$$\Delta\tau = \mathbf{E}\left[\tau^*(0) - \tau^*(\nu)\right] = \frac{\log\zeta}{\alpha - \sigma^2/2}, \tag{17}$$

where $\zeta$ is defined as in (16).

An important feature of the obtained estimates (13)–(17) is that they on the initial data of the project (the amount of the required investment $I$ and the initial value of virtual profit $\pi_0$), and are determined only by the parameters of the project $(\alpha, \sigma)$, the discount rates of the investor and environment $\mu, \rho, \eta$, and the tax holidays $\nu$.

One can see from the formulas in Theorems 1 and 2 that, when tax holidays $\nu$ increase, the critical level $\pi^*$ decreases, and the value of the discounted tax payments into the federal budget increases. Moreover, the optimal moment of investment $\tau^*$ will decrease (and with a probability of one) where the stochastic process of profits $\Pi = (\pi_t(\omega),\ t \geq 0,\ \omega \in \Omega)$. This implies that (non-expected) discounted tax payments from the project into the federal budget $\mathfrak{T}^f_\omega(\nu) = \int\limits_{\tau^*(\nu)}^{\infty} \gamma_f \pi_t(\omega) e^{-\eta t}\, dt$ increases in $\nu$ for all random events $\omega \in \Omega$. As for discounted tax payments into the regional budget $\mathfrak{T}^r_\omega(\nu) = \int\limits_{\tau^*(\nu)+\nu}^{\infty} \gamma_r \pi_t(\omega) e^{-\eta t}\, dt$, the region will have , i.e. $\mathfrak{T}^r_\omega(\nu) > \mathfrak{T}^r_\omega(0)$, only in such cases where $\tau^*(\nu) + \nu < \tau^*(0)$. The probability $P^r$ of such an event can be estimated as follows.

Using equation (6) for geometric Brownian motion $(\pi_t, t \geq 0)$ with parameters $(\alpha, \sigma)$ and an independence of Wiener increments, one can show the following formula for any Markovian moment $\tau$:

$$\pi_{\tau+t} = \pi_\tau \exp\{(\alpha - \frac{1}{2}\sigma^2)t + \sigma(w_{\tau+t} - w_\tau)\} = \pi_\tau \exp\{(\alpha - \frac{1}{2}\sigma^2)t + \sigma\tilde{w}_t\},\ t \geq 0,$$

where $(\tilde{w}_t,\ t \geq 0)$ is Wiener process independent of $\pi_\tau$.

Applying this, we can get:

$$\begin{aligned} P^r &= \mathbf{P}\{\tau^*(\nu) + \nu < \tau^*(0)\} = \mathbf{P}\{\max_{0 \leq t \leq \tau^*(\nu)+\nu} \pi_t < \pi^*(0)\} \\ &= \mathbf{P}\{\max_{\tau^*(\nu) \leq t \leq \tau^*(\nu)+\nu} \pi_t < \pi^*(0)\} = \mathbf{E}\mathbf{P}\{\max_{0 \leq t \leq \nu} \pi_{\tau^*(\nu)+t} < \pi^*(0) \left| \pi_{\tau^*(\nu)} \right.\} \\ &= \mathbf{E}\mathbf{P}\{\max_{0 \leq t \leq \nu} \pi_{\tau^*(\nu)} \exp\{(\alpha - \frac{1}{2}\sigma^2)t + \sigma\tilde{w}_t\} < \pi^*(0) \left| \pi_{\tau^*(\nu)} \right.\} \\ &= \mathbf{P}\left\{\max_{0 \leq t \leq \nu} [(\alpha - \frac{1}{2}\sigma^2)t + \sigma\tilde{w}_t] < \log\frac{\pi^*(0)}{\pi^*(\nu)}\right\}, \\ & \quad \log\frac{\pi^*(0)}{\pi^*(\nu)} = \log\frac{1 - \widehat{\gamma}(\nu)}{1 - \widehat{\gamma}(0)} = \log\zeta. \end{aligned}$$

Now, using the known formula for the distribution of the maximum of the Wiener process with linear drift (see, for example, Shiryaev et al (1994)):

$$\mathbf{P}\{\max_{0 \leq t \leq T}(\sigma w_t + at) \leq x\} = \Phi(x_-) - \exp\{\frac{1}{2}(x_+^2 - x_-^2)\}\Phi(-x_+),$$

where $x_\pm = \dfrac{x \pm aT}{\sigma\sqrt{T}}$, $\quad \Phi(x) = (2\pi)^{-1/2} and \displaystyle\int_{-\infty}^{x} e^{-y^2/2}dy$ is the Gaussian distribution function, we get for positive $\nu$ and $\sigma$:

$$P^r = \Phi(\xi_-) - \zeta^{2\alpha/\sigma^2 - 1}\Phi(-\xi_+), \tag{18}$$

where $\xi_\pm = \left[\log\zeta \pm (\alpha - \dfrac{\sigma^2}{2})\nu\right] / (\sigma\sqrt{\nu})$.

Let us point out that this probability also does not depend on the initial data of the project ($\pi_0$ and $I$). Some examples of the magnitude of this probability will be given below in Section 5.1.

### 4.3. COMPARATIVE STATICS: DEPENDENCE ON UNCERTAINTY, RISK, AND TAX EXEMPTIONS

In this Section, we will point out the general type of dependence of the main economic indicators of the model on the parameters of the project and its environment. Let us emphasise the parameters connected with the uncertainty, risk and tax exemptions (i.e. the volatility of the project $\sigma$, the rate of the political risk $\delta$, and the tax holidays $\nu$).[7]

Table 1 describes the qualitative behaviour of the economic indicators (13)–(17) as functions of $\sigma$, $\delta$ and $\nu$. Here, arrows indicate monotonicity (in the corresponding direction), sign $\frown$ means the presence of the maximum (transition from increase to decrease), and sign $\sim$ means that the qualitative behaviour does not have a definite character and can vary depending on the composition of the input parameters.

**Table 1.** Behaviour of the main indicators as functions of volatility, political risk and tax holidays

| Indicators | Volatility $\sigma$ | Political risk $\delta$ | Tax holidays $\nu$ |
|---|---|---|---|
| Investment level, $\pi^*$ | $\nearrow$ | $\nearrow$ | $\searrow$ |
| Probability of investment, $P^*$ | $\searrow$ | $\searrow$ | $\nearrow$ |
| Exp.time of invest.waiting, $\mathbf{E}\tau^*$ | $\nearrow$ | $\nearrow$ | $\searrow$ |
| Investor NPV, $\mathfrak{V}$ | $\sim$ | $\searrow$ | $\nearrow$ |
| Federal tax payments, $\mathfrak{T}^f$ | $\sim$ | $\searrow$ | $\nearrow$ |
| Regional tax payments, $\mathfrak{T}^r$ | $\sim$ | $\searrow$ | either $\searrow$, or $\frown$ |
| Influence of tax holidays on: | | | |
| – investor NPV, $\mathcal{E}^i$ | $\searrow$ | $\nearrow$ | $\nearrow$ |
| – federal budget, $\mathcal{E}^f$ | $\searrow$ | $\nearrow$ | $\nearrow$ |
| – regional budget, $\mathcal{E}^r$ | $\searrow$ | $\nearrow$ | either $\searrow$, or $\frown$ |
| Expected speed-up, $\Delta\tau$ | $\nearrow$ | $\nearrow$ | $\nearrow$ |

As one can see from the table, if the volatility of the project $\sigma$ increases, the probability of investment (if $\alpha < \frac{1}{2}\sigma^2$) falls and the expected time of investment waiting (if $\alpha > \frac{1}{2}\sigma^2$) rises, if that were not obvious intuitively. The influence of tax holidays on investors and tax payments into the budgets (in a relative sense) decreases and, as calculations in the next section show, this decrease can be very significant.

When the political risk $\delta$ is higher, the investment level $\pi^*$ also increases. Therefore, the moment of investment $\tau^*$ is delayed of the profit process (unlike the previous case of change in the parameter $\sigma$, when the process itself changed). The later entry of the investor leads to a decrease in net discounted income and to a decrease in tax payments into the budgets but, at the same time, tax holidays become more effective (they increase relative gain).

Hence, the relative influence of tax holidays increases with an increase in political risk (though the relative income of the investor and tax payments are decreased) and decreases in volatility of the project. Concerning dependence on tax holidays, we should note that almost all our inferences are consistent with the intuitive notions. We next turn our attention to the non-monotonic behaviour of the regional tax payments $\mathfrak{T}^r$ (and the corresponding regional estimate $\mathcal{E}^r$) in tax holidays. From Theorem 2 we have the following formula:

$$\mathfrak{T}^r(\nu) = \frac{\gamma_r \pi_0}{\eta - \alpha} \left( \frac{\pi_0}{k(\mu - \alpha) I} \right)^{\tilde{\beta} - 1} u(\nu),$$

where $u(\nu) = e^{-(\eta-\alpha)\nu}(1-\widehat{\gamma})^{\tilde{\beta}-1}, \ \mu = \rho + \delta$. Then

$$\begin{aligned}
u'(\nu) &= -(\eta-\alpha)e^{-(\eta-\alpha)\nu}(1-\widehat{\gamma})^{\tilde{\beta}-1} + (\tilde{\beta}-1)e^{-(\eta-\alpha)\nu}(1-\widehat{\gamma})^{\tilde{\beta}-2} \\
&\times \ \gamma_r(\mu-\alpha)e^{-(\mu-\alpha)\nu} = (1-\widehat{\gamma})^{\tilde{\beta}-2}e^{-(\eta-\alpha)\nu} \\
&\times \ \Big[(\tilde{\beta}-1)\gamma_r(\mu-\alpha)e^{-(\mu-\alpha)\nu} - (\eta-\alpha)(1-\widehat{\gamma})\Big] \\
&= (1-\widehat{\gamma}(\nu))^{\tilde{\beta}-2}(\eta-\alpha)e^{-(\eta-\alpha)\nu}\left\{\gamma_r e^{-(\mu-\alpha)\nu}\left[(\tilde{\beta}-1)\frac{\mu-\alpha}{\eta-\alpha}+1\right]\right. \\
&- \left. 1+\gamma_f \right\}.
\end{aligned}$$

Now it is easy to see that, if

$$\tilde{\beta} \le 1 + \frac{1-\gamma_f-\gamma_r}{\gamma_r} \cdot \frac{\eta-\alpha}{\mu-\alpha},$$

then $u'(\nu) \le 0$ for all $\nu \ge 0$, but if the opposite inequality

$$\tilde{\beta} > 1 + \frac{1-\gamma_f-\gamma_r}{\gamma_r} \cdot \frac{\eta-\alpha}{\mu-\alpha}, \tag{19}$$

holds, then regional tax payments have a unique maximum at the point

$$\nu^* = \frac{1}{\mu-\alpha}\log\left[\frac{\gamma_r}{1-\gamma_f}\left(1+(\tilde{\beta}-1)\frac{\mu-\alpha}{\eta-\alpha}\right)\right]. \tag{20}$$

Such "optimal" tax holidays $\nu^*$ are able to bring to the region maximum of discounted tax payments from projects (in income tax). We will investigate in detail these tax holidays in the next Section.

## 4.4. COMPARISON OF THE INFLUENCE OF DIFFERENT FACTORS ON INVESTMENT ACTIVITY

In this Section we will try to compare the influence of the parameters of the economic environment – discount $\rho$, political risk $\delta$, tax rates for the federal and regional budgets $\gamma_f$ и $\gamma_r$, as well as tax holidays $\nu$ – on investor behaviour (investment activity).

In line with Theorem 1, we can characterise an investment activity by an optimal threshold $\pi^*$ such that, when virtual profit reaches it, a decision on investment is adopted. The explicit expression in (12) shows dependence $\pi^*$ on the factors of the economic environment while the parameters of the project $(\alpha, \sigma)$ have a significantly complicated and non-linear character.

Let us calculate the partial derivatives of the function:

$$\pi^* = \pi^*(\rho, \delta, \gamma_f, \gamma_r, \nu) = I \cdot \frac{\beta}{\beta - 1} \cdot \frac{\rho + \delta - \alpha}{1 - \widehat{\gamma}},$$

where $\widehat{\gamma} = \gamma_f + \gamma_r e^{-(\rho+\delta-\alpha)\nu}$ and $\beta = \beta(\rho)$ is as defined at the beginning of Section 4.1. We have:

$$\begin{aligned}
\frac{\partial \pi^*}{\partial \gamma_f} &= I\frac{\beta}{\beta-1} \cdot \frac{\rho+\delta-\alpha}{(1-\widehat{\gamma})^2} = \frac{\pi^*}{1-\widehat{\gamma}}, && (21)\\
\frac{\partial \pi^*}{\partial \gamma_r} &= \frac{\pi^*}{1-\widehat{\gamma}} e^{-(\rho+\delta-\alpha)\nu}, && (22)\\
\frac{\partial \pi^*}{\partial \nu} &= -\frac{\pi^*}{1-\widehat{\gamma}} \gamma_r (\rho+\delta-\alpha) e^{-(\rho+\delta-\alpha)\nu}, && (23)\\
\frac{\partial \pi^*}{\partial \delta} &= I\frac{\beta}{\beta-1} \cdot \frac{1-\widehat{\gamma}-(\rho+\delta-\alpha)\gamma_r \nu e^{-(\rho+\delta-\alpha)\nu}}{(1-\widehat{\gamma})^2} \\
&= \frac{\pi^*}{1-\widehat{\gamma}} \left( \frac{1-\widehat{\gamma}}{\rho+\delta-\alpha} - \gamma_r \nu e^{-(\rho+\delta-\alpha)\nu} \right), && (24)\\
\frac{\partial \pi^*}{\partial \rho} &= -\frac{I}{(\beta-1)^2} \frac{\rho+\delta-\alpha}{1-\widehat{\gamma}} \frac{\partial \beta}{\partial \rho} + I\frac{\beta}{\beta-1} \\
&\times \quad \frac{1-\widehat{\gamma}-(\rho+\delta-\alpha)\gamma_r \nu e^{-(\rho+\delta-\alpha)\nu}}{(1-\widehat{\gamma})^2} = -\frac{\pi^*}{\beta(\beta-1)} \frac{\partial \beta}{\partial \rho} + \frac{\partial \pi^*}{\partial \delta}.
\end{aligned}$$

Differentiating both parts of the quadratic equation (11) by $\rho$ we have:

$$\frac{\partial \beta}{\partial \rho} = \frac{1}{\sigma^2(\beta - 1/2) + \alpha}.$$

For this reason:

$$\frac{\partial \pi^*}{\partial \rho} = \frac{\pi^*}{1-\widehat{\gamma}} \left( \frac{1-\widehat{\gamma}}{\rho+\delta-\alpha} - \frac{1-\widehat{\gamma}}{\beta(\beta-1)[\sigma^2(\beta-1/2)+\alpha]} - \gamma_r \nu e^{-(\rho+\delta-\alpha)\nu} \right). \tag{25}$$

We should ensure that all derivatives in formulas (22)-(25) are expressed through the derivative by the federal tax rate $\gamma_f$ (21). For this reason, in a comparison of the derivatives, it is sufficient to consider the following expressions:

$$D_r = \frac{\partial \pi^*}{\partial \gamma_r} \bigg/ \frac{\partial \pi^*}{\partial \gamma_f}, \quad D_\nu = \frac{\partial \pi^*}{\partial \nu} \bigg/ \frac{\partial \pi^*}{\partial \gamma_f},$$
$$D_\rho = \frac{\partial \pi^*}{\partial \rho} \bigg/ \frac{\partial \pi^*}{\partial \gamma_f}, \quad D_\delta = \frac{\partial \pi^*}{\partial \delta} \bigg/ \frac{\partial \pi^*}{\partial \gamma_f}.$$

These ratios of partial derivatives of the investment threshold $\pi^*$ can be considered as marginal estimates of the influence of different factors on investor activity. They show what changes in the different parameters cause the same changes in the function $\pi^*$. Indeed, if we wish to compare the federal and the regional tax rates, let us consider the following relationships:

$$\pi^*(\gamma_f + \Delta\gamma_f, \gamma_r) \approx \pi^*(\gamma_f, \gamma_r) + \frac{\partial \pi^*}{\partial \gamma_f} \Delta\gamma_f,$$

$$\pi^*(\gamma_f, \gamma_r + \Delta\gamma_r) \approx \pi^*(\gamma_f, \gamma_r) + \frac{\partial \pi^*}{\partial \gamma_r} \Delta\gamma_r$$

(other arguments in $\pi^*$ are omitted). Hence, the increment $\Delta\gamma_r$ causes the same change in $\pi^*$ (under the fixed other arguments) as the increment $\Delta\gamma_f$ provided that they are related by the equation:

$$\Delta\gamma_r = (1/D_r)\Delta\gamma_f. \tag{26}$$

Similarly, one can derive the following relationships for increments in arguments that cause the same increments in $\pi^*$:

$$\Delta\nu = (1/D_\nu)\Delta\gamma_f, \quad \Delta\delta = (D_\rho/D_\delta)\Delta\rho, \quad \Delta\nu = (D_\delta/D_\nu)\Delta\delta. \qquad (27)$$

We should point out that always $D_r < 1, \quad D_\rho < D_\delta$. The typical magnitudes of values $D_\delta, D_\rho$, etc. are given in Table 2. The calculations there are made for tax rates equalling $\gamma_f = 13\%,\ \gamma_r = 22\%$, variants of tax holidays ($\nu = 0, 3, 5$ years) which are typical for the Russian regions, as well as discount $\rho = 20\%$ (per year) and projects with parameters $\alpha = 0,\ \sigma = 0.04$ (all data are on annual basis). Note that, as one can see from equation (27), $D_r$ has a non-dimensional value, while the units of measurement for other values are: for $D_\rho,\ D_\delta$, and $^{-1}$ for $D_\nu$.

**Table 2**

| $\nu$ | Polit.risk $D_\delta$ | Discount $D_\rho$ | Region.tax $D_r$ | Tax holidays $D_\nu$ |
|---|---|---|---|---|
| ($\delta = 0$) | | | | |
| 0 | 3.25 | 3.15 | 1 | -0.044 |
| 3 | 3.38 | 3.27 | 0.55 | -0.024 |
| 5 | 3.54 | 3.42 | 0.37 | -0.016 |
| ($\delta = 0.02$) | | | | |
| 0 | 2.95 | 2.85 | 1 | -0.048 |
| 3 | 3.10 | 2.98 | 0.52 | -0.025 |
| 5 | 3.26 | 3.13 | 0.33 | -0.016 |

Note that negative $D_\nu$ means that effective increases in tax rates is equivalent to decreases in tax holidays, and vice versa.

The results of the calculations in Table 2 are robust enough with respect to changes in the parameters of the model (within "reasonable" values). So, this table, together with equations (26)–(27), allow us to make the following conclusions concerning the marginal influence of different factors on investor activity.

1) The political risk parameter $\delta$ and discount $\rho$ have an almost equal influence

on the investor (the influence of $\delta$ is a little bit stronger).

2) If the existing tax holidays are $3 \div 5$ years, an increment in the federal tax rate of $1\%$ causes the same changes in investment activity as an increment in the regional tax rate of $2 \div 3\%$. If there are no tax holidays, both rates are equivalent from the investor's point of view. An increase in tax holidays of $1$ year is equivalent to a decrease in the federal tax rate of approximately $1.5 \div 2.5\%$ (for 3- or 5-year tax holidays) and by $4 \div 5\%$ (if there are no holidays).

3) Comparing political risk and tax holidays, we can say that a decrease in political risk of $1\%$ causes the same changes in investment activity as an increase in tax holidays of $1.5 \div 2$ years (for typical tax holidays), or of $0.6 \div 0.7$ years (without tax holidays).

Until now we have been comparing different factors, or equivalently, the "compensation" of changes in one factor by another in the marginal sense, i.e. by small changes in parameters.

Let us look at the problem of compensation from the other side.

Suppose that an investor who follows the optimal behaviour strategy described in Section 2, faces a dilemma: investing in a "risky" economy, that provides significant tax holidays; or going to a "no-risk" economy, and not to have any tax exemptions (or only minimal ones). An investor can face such a problem even when choosing a region for the realisation of the project. There appears a question: which tax exemptions can compensate (from the point of view of investor entry) the political risk factors? We should point out that we mean here significant changes in parameters, not small ones as above.

As we have already pointed out, the activity of the investor within the framework of our model is characterised by the critical level for investment $\pi^*$ (see Theorem 1). Furthermore, (until the end of this Section) we will write also $\pi^* = \pi^*(\delta, \nu)$ in order to emphasise its dependence on the factors of political risk $\delta$ and tax holidays $\nu$.

We will say that tax holidays $\nu_\delta$  political risk if they do not change the optimal level of investment in comparison with the no-risk case, i.e.:

$$\pi^*(\delta, \nu_\delta) = \pi^*(0, \nu_0), \tag{28}$$

where $\nu_0$ is the tax holidays existing in the no-risk case (in particular, $\nu_0$ can be equal to zero, i.e. without risk there are no exemptions).

Assuming (for the sake of simplicity) that the parameters of the project, tax rates and discount are the same in both the risk and no-risk cases, from the explicit form for $\pi^*$ (Theorem 1) one can easily see that (28) is equivalent to the following:

$$1-\gamma_f-\gamma_r e^{-(\rho+\delta-\alpha)\nu_\delta}=\left(1+\frac{\delta}{\rho-\alpha}\right)\left(1-\gamma_f-\gamma_r e^{-(\rho-\alpha)\nu_0}\right);$$

or:

$$e^{-(\rho+\delta-\alpha)\nu_\delta}=e^{-(\rho-\alpha)\nu_0}-\frac{\delta}{\rho-\alpha}\left(\frac{1-\gamma_f}{\gamma_r}-e^{-(\rho-\alpha)\nu_0}\right). \qquad (29)$$

In order to satisfy equation (29) with some $\nu_\delta$, the following condition must be valid:

$$\frac{\rho-\alpha}{\delta}>\frac{1-\gamma_f}{\gamma_r}e^{(\rho-\alpha)\nu_0}-1>\frac{1-\gamma_f-\gamma_r}{\gamma_r}.$$

Hence, if $\delta>\gamma_r(\rho-\alpha)/(1-\gamma_f-\gamma_r)\approx 0.34(\rho-\alpha)$ under existing tax rates in Russia, then (29) can not be valid for any $\nu_\delta$.

It means that there is a "critical" value of political risk $\delta^*$ such that, if political risk is greater than this value, it  (in the sense of (28)) by any tax holidays. This "critical" value is equal to:

$$\delta^*=(\rho-\alpha)\gamma_r\Big/\Big[(1-\gamma_f)e^{(\rho-\alpha)\nu_0}-\gamma_r\Big]$$

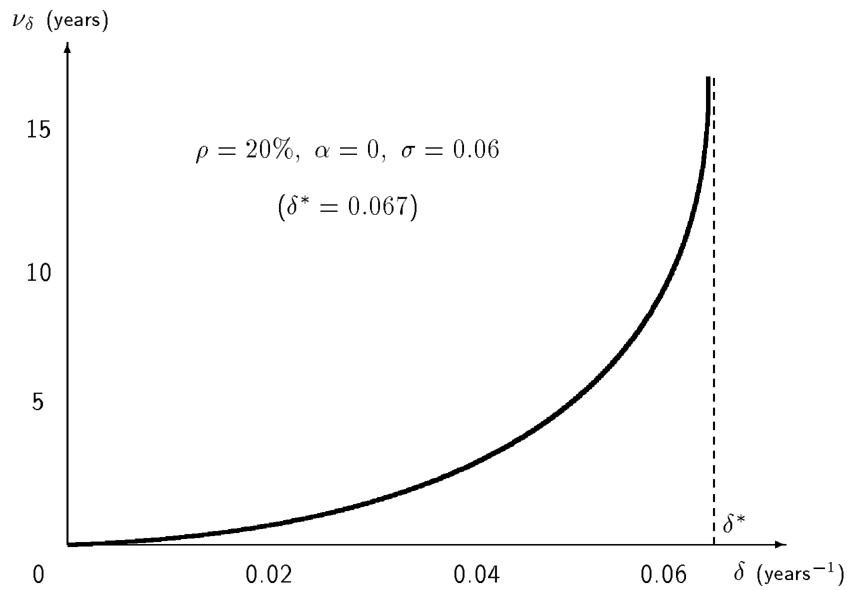
In particular, if there are no exemptions ($\nu_0=0$) in a "no-risk" economy, then $\delta^*\approx 0.34(\rho-\alpha)$ (approximately $0.05\div 0.06$ per year), and, in the general case, it depends on $\nu_0$.

The formula for the compensating tax holidays $\nu_\delta$ is the following:

$$\nu_\delta=\nu_0-\frac{1}{\rho+\delta-\alpha}\log\left[1-\frac{\delta}{\rho-\alpha}\left(\frac{1-\gamma_f}{\gamma_r}e^{(\rho+\delta-\alpha)\nu_0}-1\right)\right].$$

The typical pattern of their behaviour (for the case $\nu_0=0$) is given by Figure 1.

**Figure 1.** Dependence of the compensating tax holidays on political risk




## 5. PRINCIPLES OF THE DETERMINATION OF TAX HOLIDAYS

In previous Sections, the investment project was considered from the point of view of the investor. Now, let us look at it from another point of view — that of regional interests. Here, we will view the basic model from Section 2 as a model of investor behaviour from the point of view of the region.

As was noted previously, the creation of a new enterprise in the region associated with the appearance of a new taxpayer leads to an increase in employment, but it can also bring a number of new problems (for example, ecological ones). So, the consequences of the appearance of the investor in the region can be very different. A region can have its "own" projects in which it is very interested and ready to provide significant tax exemptions with the purpose of winning investment competitions held, for example, by the European Bank for Reconstruction and Development. At the same time, the investor who has a project for enterprise creation can choose a region for its realisation, taking into account many factors, including tax exemptions.

For these reasons, it is interesting to explore and compare different principles of the determination of tax holidays.

Here, we will analyse three principles of the determination of tax holidays. The first is the principle of the optimisation of tax payments with the help of tax holidays of appropriate duration. The other two are those principles that exist in reality — fixed tax holidays, and tax holidays during the payback period.

The basis of our consideration is the model from Section 2, i.e. the hypothesis of the optimal behaviour of the investor. If one knows the behaviour of the investor under any duration of tax holidays, the region can estimate the consequences of investor entry in addition to, in particular, tax payments into the regional budget. The above-mentioned optimisation approach gives also the possibility of evaluating the effectiveness of existing tax holidays.

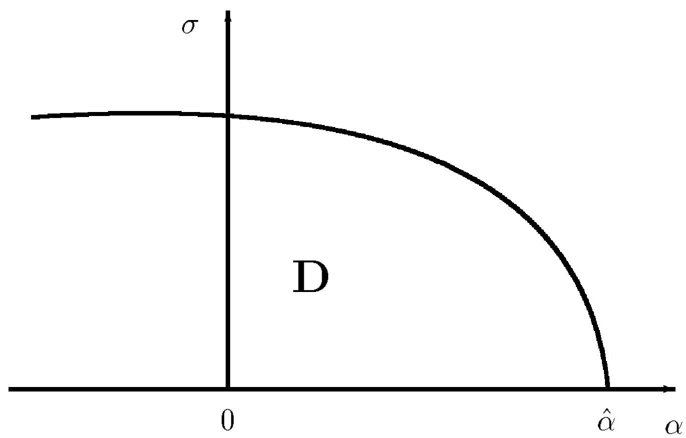
## 5.1. OPTIMISATION APPROACH

As was shown in Section 4.3, discounted tax payments from the project into the regional budget $\mathfrak{T}^r$ are maximal (over all tax holidays $\nu$) at the point $\nu = \nu^*$. This "optimal" point has an explicit form (20) when conditions (19) are satisfied, and when this condition does not hold it is equal to zero. Such tax holidays $\nu^*$ can be viewed as the "best" exemptions from the regional point of view (namely, tax payments into the regional budget from the project). Naturally, the region can have other motives for the attraction of the investor other than the purely fiscal (such as improvements in infrastructure, increase in employment, etc.). However, we will consider the concept of optimal tax holidays as a useful theoretical tool that makes it possible to compare and analyse the actual principles of the determination of tax holidays.

For this reason, we begin with a more detailed investigation of optimal tax holidays.

The set of parameters $D = \{(\alpha, \sigma)\}$, for which condition (19) holds, looks like the domain in Figure 2. From here, one can see that non-trivial tax holidays should be set only for projects for which profits have both a sufficiently small

**Figure 2.** Set of mutual benefits




expected growth rate and low volatility. In other words, for the region it does not make sense to provide tax exemptions either for projects with a high profits growth rate (large $\alpha$), or for those with high volatility (large $\sigma$).

If the profits from the project are deterministic ($\sigma = 0$), then $\tilde{\beta} = \eta/\alpha$ and condition (19) transforms to the following:

$$\mu - \alpha > \alpha \frac{1 - \gamma_f - \gamma_r}{\gamma_r}.$$

Thus, the critical growth rate for non-zero tax holidays is equal to $\hat{\alpha} = \frac{\gamma_r}{1 - \gamma_f}(\rho + \delta)$ which, under the current Russian laws ($\gamma_f = 13\%, \gamma_r = 22\%$), is about one quarter ($0.253$) of the investor discount rate including political risk. It means, in particular, that the existence of non-trivial tax holidays which maximise discounted tax payments is not obvious, even in the deterministic case, since it requires certain conditions.

From the formulas in Theorem 2, one can see that the values $\mathfrak{V}$, $\mathfrak{T}^f$ increase in $\nu$, and $\mathfrak{T}^r$ increase (in $\nu$) if $0 \leq \nu \leq \nu^*$ and decrease if $\nu > \nu^*$. Therefore, the set $D$ (defined by equation (19)), in which there are non-zero optimal tax holidays, can be termed . It means that, if project parameters are located in this set, an increase in tax holidays from $0$ as far as the optimal level $\nu^*$ is mutually beneficial for all participants because it increases both the expected discounted net income of the investor and the expected tax payments into both the regional and the federal budgets.

Let us demonstrate now the effectiveness of the mechanism of tax holidays with some numeric examples. As reasonable sets of the parameter values, we will consider $\alpha \sim -2\% \div 3\%$, $\sigma \sim 0 \div 0.1$, discount rates $\sim 10 \div 25\%$ (annual).[8]

Table 3 shows the dependence of the estimates of the effectiveness of the participants $(\mathcal{E}^f, \mathcal{E}^r, \mathcal{E}^i)$ and the expected speed-up in investment ($\Delta\tau$, in years) on the expected growth rate in profits $(\alpha)$. Also given are the values of optimal tax holidays ($\nu^*$, in years) and the probability of the *real advantage* of the region from them $(P^r)$. The volatility of the project is taken as $\sigma = 0.04$ (in this case, non-zero optimal tax holidays will be if $\alpha < 4.8\%$), all discount rates are equal to 20%, and political risk is absent $(\delta = 0)$.

**Table 3.** Dependence on expected growth rate

| Exp.rate $\alpha$ | Fed.budget $\mathcal{E}^f$ | Reg.budget $\mathcal{E}^r$ | Investor $\mathcal{E}^i$ | Speed-up $\Delta\tau$ | Holidays $\nu^*$ | Probab. $P^r$ |
|---|---|---|---|---|---|---|
| 1% | 7.27 | 2.59 | 8.85 | 6.74 | 5.42 | |
| 2% | 2.89 | 1.45 | 3.38 | 5.32 | 3.81 | 0.77 |
| 3% | 1.67 | 1.12 | 1.86 | 3.62 | 2.35 | 0.62 |

A decrease in the effectiveness of tax holidays in $\alpha$ means (in some sense) that a high rate of profits growth is a better stimulus for the investor than tax exemptions.

The next table illustrates the dependence of the same indicators on volatility $\sigma$. Expected growth rate is equal to 3%, discounts – 20%, and political risk is absent (non-zero tax holidays will be in this case if $\sigma < 0.118$).

**Table 4.** Dependence on volatility

| Volatility $\sigma$ | Fed.budget $\mathcal{E}^f$ | Reg.budget $\mathcal{E}^r$ | Investor $\mathcal{E}^i$ | Speed-up $\Delta\tau$ | Holidays $\nu^*$ | Probab. $P^r$ |
|---|---|---|---|---|---|---|
| 0 | 2.08 | 1.23 | 2.36 | 4.23 | 3.07 | 1 |
| 0.02 | 1.95 | 1.19 | 2.20 | 4.11 | 2.86 | 0.82 |
| 0.04 | 1.67 | 1.12 | 1.86 | 3.62 | 2.35 | 0.62 |
| 0.06 | 1.42 | 1.06 | 1.54 | 2.92 | 1.72 | 0.49 |
| 0.08 | 1.23 | 1.02 | 1.30 | 2.07 | 1.09 | 0.40 |
| 0.10 | 1.09 | 1.01 | 1.12 | 1.08 | 0.49 | 0.26 |

From this table it follows that the effectiveness of tax exemptions decreases if uncertainty (volatility) increases for all participants of the model. For the investor, it has to do with the fact that, if the project's profits are too volatile (and random oscillations can be both positive and negative), tax exemptions are no longer attractive (when volatility is too high, the project is outside the set of mutual benefits $D$ and there is no tax exemptions at all).

Table 5 demonstrates the dependence of the above-mentioned indicators of effectiveness both on the discount rates and political risk. Here, a project with "average" parameters $\alpha = 3\%,\ \sigma = 0.06$ is considered.

**Table 5.** Dependence on political risk and discounts

| Political risk $\delta$ (years$^{-1}$) | Fed.budget $\mathcal{E}^f$ | Reg.budget $\mathcal{E}^r$ | Investor $\mathcal{E}^i$ | Speed-up $\Delta\tau$ | Holidays $\nu^*$ | Probab. $P^r$ |
|---|---|---|---|---|---|---|
| | | | $(\rho = 15\%,$ | $\eta = 20\%)$ | | |
| 0 | 1.03 | 1.00 | 1.03 | 0.24 | 0.17 | 0.18 |
| 0.02 | 1.20 | 1.02 | 1.20 | 1.52 | 0.99 | 0.37 |
| 0.04 | 1.35 | 1.05 | 1.35 | 2.51 | 1.52 | 0.46 |
| 0.06 | 1.49 | 1.08 | 1.48 | 3.29 | 1.89 | 0.52 |
| 0.08 | 1.61 | 1.12 | 1.59 | 3.94 | 2.13 | 0.57 |
| | | | $(\rho = 20\%,$ | $\eta = 20\%)$ | | |
| 0 | 1.42 | 1.06 | 1.54 | 2.92 | 1.72 | 0.49 |
| 0.02 | 1.55 | 1.10 | 1.72 | 3.63 | 2.02 | 0.55 |
| 0.04 | 1.67 | 1.14 | 1.88 | 4.22 | 2.23 | 0.60 |
| 0.06 | 1.77 | 1.18 | 2.02 | 4.71 | 2.37 | 0.64 |
| 0.08 | 1.86 | 1.23 | 2.15 | 5.13 | 2.46 | 0.68 |
| | | | $(\rho = 20\%,$ | $\eta = 25\%)$ | | |
| 0 | 1.48 | 1.06 | 1.48 | 2.61 | 1.51 | 0.48 |
| 0.02 | 1.65 | 1.11 | 1.65 | 3.35 | 1.82 | 0.56 |
| 0.04 | 1.81 | 1.15 | 1.81 | 3.96 | 2.05 | 0.60 |
| 0.06 | 1.96 | 1.21 | 1.95 | 4.47 | 2.20 | 0.64 |
| 0.08 | 2.09 | 1.26 | 2.08 | 5.91 | 2.31 | 0.68 |

As one can see from the calculations, the additional benefits from the introduction of optimal tax holidays can be quite significant, and achieved with the help of relatively short tax holidays ($3 \div 4$ years, which is appropriate for some Russian regions). At the same time, it turns out that when the region tries to achieve the highest fiscal gain, it provides favours (in the sense of a relative gain) to the investor and to the federal budget. But, as we have already mentioned, the region may have non-tax benefits from the project. The sensitivity of the region's revenues to a change in both project parameters and political risk is also less than those for the investor and the region. As for the expected speed-up in investment, it can also be significant and achieve $4 \div 5$ years.

From this table, one can see that the mechanism of optimal tax exemptions becomes more effective (for all participants) in unstable systems with high levels of political risk. For example, if $\delta = 0.08$, the effectiveness of optimal tax holidays increases by $10 \div 20\%$ (for the regional budget) and by $30 \div 50\%$ (for the investor and the federal budget) in comparison with the absence of the political risk ($\delta = 0$).

Let us focus on the fact that, in almost all considered examples, the probability of the *real tax advantage* of the region $P^r$ turns out to be rather high. It is $50 \div 60\%$ (on average), and it is less sensitive to the value of political risk in comparison with the sensitivity to a change in the project parameters (expected rate of income growth and volatility). This means that in only approximately half of the cases can the region obtain real tax advantages from an investor when it uses optimal tax holidays as a tax stimulus.

The region can set the goal of finding such tax holidays under which the probability $P^r$ would be maximal. But this approach is not sufficiently effective since it does not lead to an essential increase in the probability of real tax advantages for regions in comparison with the same probability under optimal tax holidays. For example, for the data in Tables 4 and 5, the increase in probability does not exceed 10% and it decreases with the growth in probability. This fact allows us to say that optimal tax holidays, besides maximising tax payments (into the regional budget), "almost" maximises the probability of real tax advantages for the region.

We now turn our attention to the non-monotonic behaviour of the duration of optimal tax holidays in political risk. Indeed, according to (20) one can write:

$$\nu^* = \frac{\tilde{\beta} - 1}{\eta - \alpha} \cdot \frac{1}{x} \log \left( \frac{\gamma_r}{1 - \gamma_f}(1 + x) \right), \quad \text{where } x = \frac{\tilde{\beta} - 1}{\eta - \alpha}(\rho + \delta - \alpha).$$

Therefore:

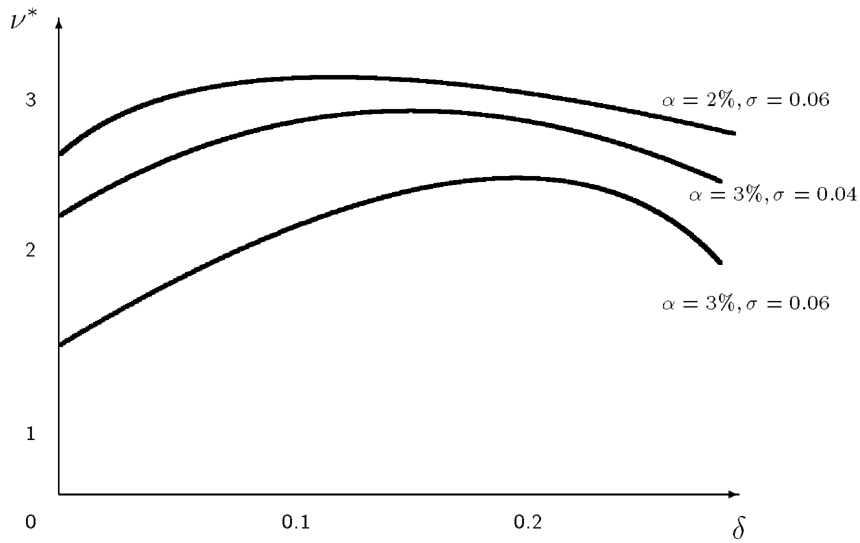
$$\frac{\partial \nu^*}{\partial \delta} = \frac{\tilde{\beta} - 1}{\eta - \alpha} \cdot \frac{\partial \nu^*}{\partial x} = \left( \frac{\tilde{\beta} - 1}{\eta - \alpha} \right)^2 x^{-2} \left[ \frac{x}{1 + x} - \log \left( \frac{\gamma_r}{1 - \gamma_f}(1 + x) \right) \right]. \tag{30}$$

Let $x^*$ be a root of the equation:

$$\log \left( \frac{\gamma_r}{1 - \gamma_f}(1 + x) \right) = \frac{x}{1 + x},$$

and $\delta^* = -\rho + \alpha + x^*(\eta - \alpha)/(\tilde{\beta} - 1)$. Then if $\delta \leq \delta^*$ the value of tax holidays $\nu^*$ increases, and if $\delta > \delta^*$, it decreases.

**Figure 3.** Dependence of optimal tax holidays on political risk




Thus tax holidays (optimal from the regional point of view) always remain bounded; after certain "critical" levels of political risk $\delta^*$ are achieved, they lose their "effectiveness" and should be decreased (Figure 3 shows several patterns of optimal tax holidays as a function of political risk for different projects for the discounts $\rho = 20\%,\ \eta = 25\%$ – see also Table 5).

### 5.2. THE PAYBACK PERIOD APPROACH (NOVGOROD SCHEME)

As was mentioned in the Introduction, in some regions of Russia the period until the break-even point is reached (the payback period) is accepted as the duration of tax holidays. During the realisation of an investment project directed to the creation of a new production process, or the reconstruction or modernisation of an existing one, tax exemptions from regional income tax are provided until the project breaks even. We will call this exemption pattern the "Novgorod scheme", since it has been most clearly manifested in the Novgorod region.

Detailed instructions for the calculation of the payback period were prepared (in the Novgorod region) by the consultancy firm Arthur Andersen. According to the accepted definitions, the payback period is determined as the minimum time interval (starting from the moment when the opening balance is recorded), during which time discounted expected cash flow (or profit) becomes equal to the initial expenditures. Depending on whether the discount factor is used in such a flow (existing instructions do not give strict recommendations on the issue), we can consider a discounted payback period, or a non-discounted payback period.

Within the framework of our model, these definitions correspond to the following values:

$\nu_1$ – the solution of the equation:

$$\mathbf{E} \int_{\tau^*}^{\tau^*+\nu_1} \pi_t dt = I; \tag{31}$$

$\nu_2$ – the solution of the equation:

$$\mathbf{E} \int_{\tau^*}^{\tau^*+\nu_2} \pi_t e^{-\rho(t-\tau^*)} dt = I. \tag{32}$$

Using obtained expressions for the optimal moment of investment, one can establish a simple relationship that makes it possible to find values $\nu_1$ and $\nu_2$.

Using formula (7) we have:

$$\begin{aligned} \mathbf{E} \int_{\tau^*}^{\tau^*+\nu_1} \pi_t dt &= \mathbf{E} \int_{\tau^*}^{\tau^*+\nu_1} \mathbf{E}(\pi_t|\mathcal{F}_{\tau^*}) dt = \mathbf{E}\pi_{\tau^*} \int_{\tau^*}^{\tau^*+\nu_1} e^{\alpha(t-\tau^*)} dt \\ &= \mathbf{E}\pi_{\tau^*} \int_0^{\nu_1} e^{\alpha t} dt = \pi^*(e^{\alpha\nu_1} - 1)/\alpha \end{aligned} \tag{33}$$

(when $\alpha = 0$ we will define the latter expression as $\pi^*\nu_1$).

For this reason, from (31) it follows that:

$$\pi^*(e^{\alpha\nu_1}-1)/\alpha = I,$$

and, by substituting the formula for $\pi^*$ from Theorem 1, we obtain the equation:

$$k\frac{\mu-\alpha}{\alpha}(e^{\alpha\nu_1}-1) = 1-\gamma_f-\gamma_r e^{-(\mu-\alpha)\nu_1}, \tag{34}$$

where $\mu = \rho+\delta,\ k = \dfrac{\beta}{\beta-1}$.

Similarly, we can obtain:

$$\mathbf{E}\int\limits_{\tau^*}^{\tau^*+\nu_2} \pi_t e^{-\rho(t-\tau^*)}dt = \mathbf{E}\pi_{\tau^*}\int\limits_0^{\nu_2} e^{-(\rho-\alpha)t}dt = \pi^*[1-e^{-(\rho-\alpha)\nu_2}]/(\rho-\alpha)$$

and the relationship:

$$k\frac{\mu-\alpha}{\rho-\alpha}\left[1-e^{-(\rho-\alpha)\nu_2}\right] = 1-\gamma_f-\gamma_r e^{-(\mu-\alpha)\nu_2}. \tag{35}$$

Thus, payback periods $\nu_1$ and $\nu_2$ can be found as the roots of equations (34) and (35).

If $\delta = 0$, then equation (35) has the explicit solution:

$$\nu_2 = \frac{1}{\rho-\alpha}\log\frac{k-\gamma_r}{k-1+\gamma_f}.$$

In Table 6, we provide some numerical calculations of payback periods $\nu_1$ и $\nu_2$ and compare them with "optimal" tax holidays $\nu^*$ both in duration and effectiveness (here effectiveness $E(\nu)$ of tax holidays $\nu$ is defined as the ratio of the tax payments into the regional budget under holidays $\nu$ to those tax payments under optimal holidays $\nu^*$). We divide all investment projects into three groups depending on their volatility: with high ($\sigma = 0.10$), moderate ($\sigma = 0.04$), and low ($\sigma = 0.01$) volatilities. We assume also $\rho = 20\%,\ \delta = 0$.

**Table 6.** Effectiveness of payback periods for groups of projects

| $\alpha$ | $\nu_1$ | $E(\nu_1)$ | $\nu_2$ | $E(\nu_2)$ | $\nu^*$ |
|---|---|---|---|---|---|
| $(\sigma = 0.10)$ | | | | | |
| -3% | 3.1 | 0.955 | 5.8 | 0.947 | 4.3 |
| -2% | 3.2 | 0.986 | 5.8 | 0.918 | 3.9 |
| -1% | 3.2 | 0.999 | 5.8 | 0.884 | 3.4 |
| 0% | 3.2 | 0.995 | 5.7 | 0.847 | 2.7 |
| 1% | 3.2 | 0.974 | 5.6 | 0.808 | 2.1 |
| 2% | 3.2 | 0.939 | 5.5 | 0.770 | 1.3 |
| 3% | 3.2 | 0.896 | 5.4 | 0.733 | 0.5 |
| $(\sigma = 0.04)$ | | | | | |
| -3% | 3.4 | 0.071 | 7.2 | 0.678 | 10.5 |
| -2% | 3.5 | 0.185 | 7.4 | 0.846 | 9.7 |
| -1% | 3.6 | 0.408 | 7.4 | 0.967 | 8.6 |
| 0% | 3.6 | 0.707 | 7.3 | 0.999 | 7.1 |
| 1% | 3.6 | 0.924 | 7.1 | 0.953 | 5.4 |
| 2% | 3.5 | 0.998 | 6.7 | 0.880 | 3.8 |
| 3% | 3.5 | 0.980 | 6.4 | 0.810 | 2.4 |
| $(\sigma = 0.01)$ | | | | | |
| -1% | 3.7 | 0.000 | 8.4 | 0.002 | 19.1 |
| 0% | 3.8 | 0.008 | 8.5 | 0.417 | 13.8 |
| 1% | 3.7 | 0.598 | 7.9 | 0.999 | 8.1 |
| 1.5% | 3.7 | 0.848 | 7.5 | 0.977 | 6.4 |
| 2% | 3.6 | 0.961 | 7.2 | 0.929 | 5.0 |
| 3% | 3.6 | 0.995 | 6.7 | 0.838 | 3.0 |

The analysis of the obtained results makes it possible to come to the following conclusions:

1) The non-discounted payback period $\nu_1$ shows significant robustness to the parameters of the project, and changes very insignificantly within 3–4 years. The discounted payback period $\nu_2$ is also robust, and is greater than $\nu_1$ by a factor of two.

2) Values $\nu_1$ and $\nu_2$ decrease both with an increase in the expected rate of growth of income from the project (such is intuitively obvious) and any decrease in the volatility of the project (uncertainty).

3) As for comparison with the optimal tax holiday $\nu^*$ (from the point of view of the region), it is impossible to prefer any of the payback period variants (with discount or without it). When the growth rate or the volatility of the project is low enough, the discounted payback period $\nu_2$ is closer to the optimal both in duration and effectiveness. When the parameters of the project $\alpha$ и $\sigma$ increase, optimal holidays decrease, and converge with the non-discounted payback period $\nu_1$. It means that any variant of the calculation of the payback period (with discount or without it) can be used under certain circumstances.

## 5.3. FIXED TAX HOLIDAYS

As was noted above, the most popular form of tax holidays in Russia is, at present, exemption from payment of regional income tax for some fixed period (after the first balance is obtained) for all enterprises on the territory of the region (which fulfil some additional conditions such as type of industry, share of foreign capital, etc.). The duration of such exemptions is usually 3–5 years.[9]

The exemptions themselves can be total, partial (for example, for 50%) or according to some time schedule (see Table 1). Unlike the holidays considered in the previous two Sections, these holidays do not depend on the individual parameters of enterprises or of investment projects, although their duration and form of organisation (type of exemption) can be changed depending on the direction and priority of the investment projects. In the future, we will consider total exemption from tax during such holidays, since "partial" holidays can be recalculated into "total" holidays with shorter duration (this will not be discussed in detail here).

The results of the calculations from Table 7 make it possible to compare such "fixed" holidays with the optimal $\nu^*$. In this table, for three groups of projects (similar to those in Table 6) we give values for optimal tax holidays $\nu^*$, as well as the indicators of effectiveness of three- and five- year tax holidays for the regions, i.e. the ratios of maximum tax payments into the regional budget (under optimal tax holidays $\nu^*$) — $E(3)$ и $E(5)$ correspondingly.

**Table 7.** Effectiveness of fixed tax holidays

| $\alpha$ | $\nu^*$ | $E(3)$ | $E(5)$ |
|---|---|---|---|
| $(\sigma = 0.10)$ | | | |
| -3% | 4.3 | 0.942 | 0.988 |
| -2% | 3.9 | 0.977 | 0.970 |
| -1% | 3.4 | 0.997 | 0.942 |
| 0% | 2.7 | 0.998 | 0.905 |
| 1% | 2.1 | 0.982 | 0.860 |
| 2% | 1.3 | 0.951 | 0.813 |
| 3% | 0.5 | 0.908 | 0.763 |
| $(\sigma = 0.04)$ | | | |
| -3% | 10.5 | 0.043 | 0.259 |
| -2% | 9.7 | 0.124 | 0.434 |
| -1% | 8.6 | 0.310 | 0.672 |
| 0% | 7.1 | 0.608 | 0.897 |
| 1% | 5.4 | 0.868 | 0.996 |
| 2% | 3.8 | 0.987 | 0.976 |
| 3% | 2.4 | 0.993 | 0.904 |
| $(\sigma = 0.01)$ | | | |
| -1% | 19.1 | 0.000 | 0.000 |
| 0% | 13.8 | 0.002 | 0.038 |
| 1% | 8.1 | 0.475 | 0.794 |
| 1.5% | 6.4 | 0.761 | 0.963 |
| 2% | 5.0 | 0.916 | 0.999 |
| 3% | 3.0 | 0.999 | 0.944 |

Tables 6 and 7 show that, for projects with a high level of volatility, optimal tax holidays (as well as both variants of the payback period) are within 2–5 years (except for highly dynamic projects with income growth rates of more than 1% which need very few incentives). The effectiveness of 3- and 5-year tax holidays is very high, so that tax revenues for 3-year holidays are different from the optimal by not more than 5%.

For projects with low volatility, a specific feature is the lengthy duration of optimal tax holidays, which significantly exceed 5 years (except in cases of large income growth rates). Therefore, the effectiveness of 5-year holidays remains low for moderate growth rates (in the order of 1%) and are "almost zero" for projects with low dynamics (where $\alpha$ is about 0 or less).

Projects with moderate levels of volatility lie between the above two extremes. Optimal tax holidays are close to 5 years, although they can be slightly longer. The effectiveness of 5-year holidays is quite high, except only for negative growth rates $\alpha$.

## 6. CONCLUSIONS AND FINAL REMARKS

1. The consideration of risk and uncertainty factors during the estimation of the effectiveness of investment projects is difficult to analyse even in the stable economies, let alone in Russia. At the same time, it is obvious that these factors play a very important role in investment decisions. The official methods (Metodicheskie rekomendatsii..., 1994) recommend calculating the effectiveness of the project under different scenarios for changes in the economic environment. The choice of such scenarios is made with expert methods and often depends on subjective opinions.

The proposed model can be viewed as a highly aggregated description of investor behaviour in the economic environment, which is subjected to different stochastic fluctuations and has certain "extreme properties". Our main hypothesis is an assumption of the distribution of the profits of investment projects (geometric Brownian motion), which reflects the element of unpredictability (the chaotic character) of small changes in income, along with their exponential growth or fall. Such a process is characterised only by two parameters, which have clear economic sense: the expected instantaneous rate of income growth and its variance (volatility, rate of uncertainty). These parameters can be evaluated on the basis of known statistical methods and regression analysis applied for observed virtual profits.

The most restrictive element of our hypothesis is the requirement for non-negative revenues after the investment, but in the first place it can be relaxed, for example with the help of the introduction of a lag period between the moment of investment and production of profits[10] , and in the second place, there are now many projects (for example, in the energy sector or in the revival of "frozen" lines of technology) which will bring profits immediately (or within a short period of time) after the investment.

2. In the proposed model of investor behaviour, it was possible to obtain

an explicit solution (in analytical form) of the investor problem, namely, to find the investment rule that maximises the investor's net present income from the project. The optimal moment for investment was used to obtain explicit formulas for such economic indicators as discounted investor revenue, discounted tax payments into the regional and federal budgets, expected speed-up of investment, and so on. Using obtained formulas, one can make both quantitative calculations, and a theoretical analysis of the dependence of the above-mentioned indicators on the parameters of the project and its environment.

3. It is shown that investment activity (which characterises investor entry) decreases if either the political risk or the volatility of the project increase. At the same time, the effectiveness of the mechanism of tax holidays (the ratio of participants' revenues under optimal tax exemptions to revenues when there are no exemptions) increases either if political risk increases (in unstable systems) or if the volatility of the project decreases.

4. A comparison was conducted of the influence on investment activity of different factors (such as tax rates and holidays, discount, and the rate of political risk). It is shown that there exists a "critical" level for political risk such that, if risk exceeds this level, it can not be compensated for (compared with no-risk cases) by any tax exemptions (tax holidays).

5. The principle was proposed of the determination of "optimal" tax holidays, which give maximum of expected discounted tax payments (over all tax holidays) into the regional budget. It turned out that, for a wide range of parameters, optimal tax holidays provide "almost" maximal probability of real tax advantages for the region.

6. We have found a set of values for the parameters of the project and its environment (the set of mutual benefits), in which an increase in tax exemptions is beneficial to all participants, since it leads to an increase in their revenues. As numerical calculations (using adjusted real data) show, if tax holidays are short enough (approximately $3 \div 4$ years), a significant increase (by several times) can be achieved in the investor's NPV, tax payments into the federal budget, and (to a smaller degree) tax payments into the regional budget. The expected speed-up of investment is increased approximately by $4 \div 5$ years.

It was found out that the effectiveness of the mechanism of optimal tax holidays increases if the political risk increases (in unstable systems) and decreases if the volatility of the project increases. For example, a growth of effectiveness (in comparison with the absence of political risk) can be $10 \div 20\%$ (for the regional tax payments) and $30 \div 50\%$ (for the investor and federal tax payments) and the expected speed-up of the investment can realise $4 \div 5$ years.

7. We compared the proposed "optimal" principle of the determination of tax holidays with the real ones existing in Russian regions – i.e. "fixed tax holidays (usually for 3–5 years)" and "payback period" principles. It was shown that real tax holidays are good enough (in duration and effectiveness in comparison with optimal tax holidays) only for investment projects with either a rather high volatility or a quite large expected income growth rate. Moreover, there does not exist a single "real" principle which would be good for all groups of projects. For investment projects with low expected income growth rates and insufficiently high volatility, the optimal tax holidays are significantly longer than those currently existing in reality, and it can increase considerably tax payments into the regional budget.

## APPENDIX A. MAIN REGIONAL TAX EXEMPTIONS FOR INVESTORS IN RUSSIA

The data are taken on the basis of the computerised information service and publications in .

| | |
|---|---|
| Vologda Region | Exemptions from income tax for priority sectors. Decrease in tax base. |
| Komi Republic | Decrease in income tax to 50-90% during payback period, property tax – 0.01%. Tax credits. |
| St.Petersburg | Decreased income tax rate (20%).<br>Exemptions from income tax (decreased tax base), property tax, housing tax, tax and rent credits. |
| Novgorod Region | 1.Productive enterprises with foreign investments are exempt from all regional taxes during the payback period.<br>2. In this region, there are four tax-free districts in which enterprises are exempt from all regional and local taxes, and also they are reimbursed for the federal income tax paid. |
| Pskov Region | Decrease in the income tax rate by 50% for 3 years (can be extended to 10 years) for investments in sectors that have priority in the region, and for enterprises engaged in production for export (more than 50% of total volume), or involved in leasing operations. |
| Vladimir Region | Production enterprises with foreign investments are exempt from income tax for 2 years, and under certain conditions can pay 25% and 50% of income tax during the third and fourth years. |
| Moscow | Investment expenditures are substracted from income tax (up to 25% of income taxes). |

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| Moscow Region | For investments in production enterprises (more than \$1m) which have a payback period of less than 3 years:<br>1. For 1.5 years, the income tax rate is 10% and for the next year – 15%.<br>2. Property tax is decreased by 50%. |
| Bryansk Region | 1. Exemptions for investment projects (for investments from other regions) concerning income, property and land taxes according to the following rates:<br>1996 – 100%, 1997 – 80%, 1998 – 70%,<br>1999 – 60%, 2000 – 50%.<br>2. Joint ventures registered after January 1, 1996, where foreign investors have more than 30% of the equity and have invested more than \$30,000, are exempt from income and property taxes according to the following rates:<br>first two years – 100%, third year – 75%,<br>fourth year – 50%.<br>3. Full tax exemption for new enterprises, and for the reconstruction and modernisation of existing ones, during the payback period but for no longer than 3 years. The property rate for such enterprises is 0.2%. Exemptions can be extended up to 10 years. |
| Ryazan Region | For production enterprises with foreign investments of more than \$30,000, exemptions on income, property and education taxes according to the following rates:<br>first two years – 100%; third year – 75%;<br>fourth year – 50%. |
| Tver' Region | 1. Decrease in income tax by 50% for up to 3 years (this period can be extended up to 10 years).<br>2. For leasing companies – decrease in the rate of income tax by 90% during the first 3 years and by 50% during the next 2.<br>3. The property tax rate is 0.2% for up to 3 years. |
| Yaroslavl Region | 1. Income tax exemptions of 100% during the first year.<br>2. Property tax is decreased by 50% (for investment projects). |

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| Nizhni Novgorod Region | 1. Decreased income tax rate (21%).<br>2. Decreased income tax rate for enterprises that pay taxes on time (up to 17%). |
| Kirov Region | For direct foreign investments (greater than $30,000) income tax is decreased:<br>for the first two years – by 100%;<br>for the third and fourth years – by 90%. |
| Republic of Mari El | For investments exceeding $100,000, exemptions on income tax, property tax, value added tax, and transportation tax according to the following rates:<br>first two years – 100%; third year – 75%;<br>fourth year – 50%; fifth year – 25%.<br>For investments of less than $100,000 — exemptions from income taxes, property tax, value added tax and transportation tax according to the following rates:<br>first two years – 50%; next two years – 25%. |
| Republic of Mordovia | For investment projects with foreign investments (greater than $3m or more than 30% of shareholders' equity) for 3-5 years:<br>- income tax exemptions of 50<br>- property tax Ц a rate of 0.05%;<br>- exemption from land tax. |
| Republic of Chuvashia | 1. New Russian enterprises and enterprises with foreign investment are exempt from all regional taxes (except property tax) for 3 years.<br>2. Income tax is decreased to 5%, if the share of the foreign partner is more than 70% and greater than $100,000. |
| Belgorod Region | Exemptions from income, property and land taxes according to the rates:<br>first year of start-up – 100%;<br>second year – 80%; third year – 60%. |
| Voronezh Region | Exemptions from tax and rent payments for a certain period, decrease in tax rates, tax credits for investments in priority sectors. |
| Lipetsk Region | Exemptions from income and property taxes for two years, and during the next two years taxes are decreased by 50%. |

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| Republic of Tatarstan | 1. Decreased income tax rate (19%).<br>2. For new enterprises, exemptions on income tax according to the following rates:<br>first two years – 100%; third year – 75%;<br>fourth year – 50%. |
| Volgograd Region | 1. Decrease in tax rates by 50% for new enterprises, or for the reconstruction or modernisation of the existing ones, for up to 2 years (exemptions can be extended up to 10 years).<br>2. Property tax rate of 1.0% for up to 3 years.<br>3. Tax investment credits for up to 2 years. |
| Ulyanovsk Region | Exemptions from income, property and value added taxes for 5 years. |
| Stavropol Territory | 1. Income tax exemptions of 50% during 2-3 years.<br>2. Income tax is decreased by 50% during the payback period (but for no longer than 3 years) for the reconstruction, modernisation or creation of new production facilities.<br>3. Decrease in property tax by 50%.<br>4. Tax credits for 5 years. |
| Republic of Bashkortostan | Small enterprises in priority sectors are exempt from income tax according to the following rates:<br>first two years – 100%, third year – 75%,<br>fourth year - 50%. |
| Udmurt Republic | 1. Income and property tax exemptions for joint ventures, where foreign investors have more than 30% of the equity and have invested more than $30,000, according to the following rates:<br>first two years – 100%;<br>third year – 50%; fourth year - 25%.<br>2. Full tax exemption for new enterprises, or for the reconstruction and modernisation of existing ones, during the payback period. |
| Perm' Region | 1. Decreased income tax rate (17.5%).<br>2. Income tax exemptions (50%) for up to 3 years.<br>3. Income tax exemptions and accelerated depreciation for small enterprises. |

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| Altai Territory, Free Economic Zone "Altai" | 1.Exemption from all taxes, payments and fees paid into the regional budget for 5 years.<br>2. Exemptions from income taxes and value added tax for 10 years (concerning the construction and exploitation of priority objects only). During the next 5 years not more than 50% of income taxes and value added tax are paid. |
| Omsk Region | Decreased income tax rates (up to 13-19%) for some enterprises and organisations. |
| Republic of Buryatia | 1. Tax holidays for 3 years for direct foreign investments of more than \$5m; for other investments – for 2 years.<br>2. Decrease in tax payments by 10-15% of new equipment costs. |
| Irkutsk Region | Decreased income tax rates (15%) for some enterprises. |
| Primorski Territory, Free Economic Zone "Nakhodka" | Regional income tax exemption for 5 years. |
| Khabarovsk Territory | 1. Income tax exemptions for 2 years for enterprises with foreign investments (more than 30%) and subsidiaries of foreign enterprises.<br>2. Income tax exemptions for 3 years for new enterprises with foreign investments of more than \$1m, that are used for the mining of useful minerals, and for the construction of the transport infrastructure. |
| Amur Region | 1. Exemption from income tax during the first year after the beginning of the selling of production.<br>2. Decrease in property tax for 2 years.<br>3. Investment tax credits for 3-4 years for production enterprises. |
| Sakhalin Region | Exemption from income tax for 5 years for enterprises with foreign investments. |
| Kaliningrad Region | 1. Income tax credits of up to 100% for 4-5 years.<br>2. Decreased tax rate (18%) for conversion and priority enterprises. |

## APPENDIX B. PROOFS OF THEOREMS

Denote $F(\pi) = \sup\limits_{\tau \in \mathcal{M}} \mathbf{E}^{\pi} G(\pi_\tau) e^{-\rho\tau}$ [11], where $G(\pi) = g\pi - I,\ g = \dfrac{1-\widehat{\gamma}}{\mu - \alpha}$, and supremum is taken over the set of all Markovian moments $\mathcal{M}$. [12]

The proof will consist of two stages.

Firstly, we show that, in finding the optimal stopping time for problem (2), it is sufficient to restrict our considerations only to the first exit time over some level by the process $\pi_t$, i.e. that

$$F(\pi) = \sup_{z \geq 0} \mathbf{E}^{\pi} G(\pi_{\tau_z}) e^{-\rho\tau_z}, \tag{B1}$$

where $\tau_z = \min\{t \geq 0 :\ \pi_t \geq z\}$ – is the first exit time over level $z$ by process $\pi_t$. Note that statement (B1) is used, as a rule, without any arguments by most authors (see McDonald and Siegel (1986), Dixit and Pindyck (1994)), while it is not obvious and needs strict justification because an arbitrary Markovian moment can not be represented in general as a first exit time from a particular set.

For proving (B1), we can define the operator $\Gamma q(\pi) = \sup\limits_{t \geq 0} \mathbf{E}^{\pi} q(\pi_t) e^{-\rho t}$. Show that this operator maps convex functions into convex. Indeed, let $q(\pi)$ be a convex function, and $Q_t(\pi) = \mathbf{E}^{\pi} q(\pi_t)$. Then, using representation (6) for $\pi_t$, we have (here $\tilde{\alpha} = \alpha - \sigma^2/2$)

$$\begin{aligned} Q_t\left(\frac{\pi' + \pi''}{2}\right) &= \mathbf{E}q\left(\frac{\pi' + \pi''}{2} e^{\tilde{\alpha}t + \sigma w_t}\right) \leq \frac{1}{2}\mathbf{E}q\left(\pi' e^{\tilde{\alpha}t + \sigma w_t}\right) \\ &+ \frac{1}{2}\mathbf{E}q\left(\pi'' e^{\tilde{\alpha}t + \sigma w_t}\right) = \frac{1}{2}Q_t(\pi') + \frac{1}{2}Q_t(\pi''). \end{aligned}$$

Therefore, $\Gamma q(\pi)$ will be convex as the supremum of the family of convex functions $\{Q_t(\pi)e^{-\rho t},\ t \geq 0\}$. Hence, due to the relationship:

$$F(\pi) = \lim_{N \to \infty} \Gamma^N G(\pi), \tag{B2}$$

where $\Gamma^N$ is the $N$-th degree of operator $\Gamma$ (see, for example, Shiryaev (1969), Chapter 3), the function $F(\pi)$ will also be convex.

Moreover, it is easy to see that

$$\begin{aligned}\Gamma G(\pi) &= \sup_{t\ge 0} e^{-\rho t}\mathbf{E}^{\pi}(g\pi_t - I) = \sup_{t\ge 0} e^{-\rho t}(g\pi e^{\alpha t} - I)\\ &= \begin{cases} G(\pi), & \text{if } \alpha\le 0,\ \pi>0\\ G(\pi), & \text{if } \alpha>0,\ \pi>\pi^1\\ C\cdot\pi^{\rho/\alpha}, & \text{if } \alpha>0,\ \pi\le\pi^1\end{cases},\end{aligned}$$

where $C = \alpha\left(\frac{\rho-\alpha}{I}\right)^{\rho/\alpha-1}\left(\frac{g}{\rho}\right)^{\rho/\alpha}$, $\pi^1 = I\frac{\rho}{g(\rho-\alpha)}$. It means that $\Gamma G(\pi) = G(\pi)$ for $\pi > \pi^1$. Using similar arguments, one can prove that there are real numbers $\{\pi^N,\ N = 2,3,\ldots\}$ such that $\Gamma^N G(\pi) = G(\pi)$ for $\pi > \pi^N$ (without loss of generality, we can consider that $\pi^N$ increase in $N$). So, from (B2), it follows that $F(\pi) = G(\pi)$ for all sufficiently large $\pi > \bar{\pi}$ (we can not except, in general, the case $\bar{\pi} = \infty$). According to general theory, the first arrival time at the set $D = \{\pi \ge 0 : \ F(\pi) = G(\pi)\}$ by the process $\pi_t$ will be an optimal stopping time for problem (2) (see Shiryaev (1969), Chapter 3). Our previous considerations imply that this set $D$ is an unbounded semi-interval $\{\pi :\ z \le \pi \le \infty\}$ for some $z$. This proves (B1).

The second stage is to obtain a formula for the optimal stopping time $\tau^*(\nu)$. For this purpose, the "smooth pasting" method for differential equations with free boundaries is commonly used (McDonald and Siegel (1986), Dixit and Pindyck (1994)). [13] But the strict justification of the fact that the "smooth pasting" condition leads to an optimal solution requires rather developed and complicated tools (as in Shiryaev, et al. (1994)). Here, we use another approach which seems to us more straightforward and simple.

We need the following:

**Lemma.** $\pi_t \tau_z = \min\{t\ge 0 : \pi_t \ge z\} z > \pi_0 \pi_t \theta > 0$

$$\mathbf{E}e^{-\theta\tau_z} = \left(\frac{\pi_0}{z}\right)^{\beta(\theta)}, \qquad (B3)$$

$\beta(\theta)$

$$\frac{1}{2}\sigma^2\beta(\beta-1) + \alpha\beta - \theta = 0.$$

From the theory of boundary problems for diffusion processes, it follows that the function $v(\pi) = \mathbf{E}^{\pi} e^{-\theta\tau_z}$ will be the solution to the following differential equation:

$$\theta v(\pi) = \alpha\pi v'(\pi) + \frac{1}{2}\sigma^2\pi^2 v''(\pi) \qquad (0 < \pi < z) \qquad (B4)$$

with the boundary condition $v(z) = 1$ (see Gikhman and Skorokhod (1977), Chapter 8).

The general solution to equation (B4) can be written as $v(\pi) = C_1\pi^{\beta_1} + C_2\pi^{\beta_2}$, where $\beta_1$ is a positive root, and $\beta_2$ is a negative root of the equation $\frac{1}{2}\sigma^2\beta(\beta-1)+\alpha\beta-\theta = 0$. Since for $\pi = 0$ the process is trivial - $\pi_t = 0 \quad \forall t$ - then, for this case, $\tau_z = \infty$ and, therefore, $v(0) = 0$, and consequently $C_2 = 0$. Establishing $C_1$ from the boundary condition $v(z) = 1$, we get formula (B3). Q.E.D.

Applying this Lemma we have: if $z > \pi_0$, then

$$\mathbf{E}^{\pi} G(\pi_{\tau_z}) e^{-\rho\tau_z} = \mathbf{E}^{\pi}(g\pi_{\tau_z} - I)e^{-\rho\tau_z} = (gz - I)\left(\frac{\pi}{z}\right)^{\beta}$$

Since $\tau_z = 0$, whenever $\pi \leq z$, then, according to equation (B1):

$$F(\pi) = \max\{g\pi - I, \ \sup_{z \geq \pi}(gz - I)(\pi/z)^{\beta}\}. \qquad (B5)$$

Taking a derivative of the function $f(z) = (gz - I)z^{-\beta}$, we have $f'(z) = 0$ at the point $z = z^* = \dfrac{I\beta}{g(\beta-1)}$. Moreover, $f''(z^*) = -I\beta(z^*)^{-\beta-2} < 0$. This means that the last supremum in formula (B5) is achieved at the point $z^* = \dfrac{I\beta}{g(\beta-1)}$ and the corresponding optimal stopping time will be $\tau^* = \min\{t \geq 0 : \ \pi_t \geq z^*\}$ (note that $\tau^* = 0$ whenever $\pi \leq z^*$). This completes the proof of Theorem 1.

Using relationships (8)–(10) as well as formula (B3), we have:

$$\mathfrak{V}(\nu) \quad = \quad \mathbf{E}\left(\frac{1-\widehat{\gamma}}{\mu-\alpha}\pi_{\tau^*} - I\right)e^{-\rho\tau^*} = \frac{1-\widehat{\gamma}}{\mu-\alpha}\pi^*\left(\frac{\pi_0}{\pi^*}\right)^{\beta} - I\left(\frac{\pi_0}{\pi^*}\right)^{\beta}$$

$$
\begin{aligned}
&= (k-1)I\left(\frac{\pi_0}{\pi^*}\right)^{\beta}, \\
\mathfrak{T}^f &= \frac{\gamma_f}{\eta-\alpha}\mathbf{E}\pi_{\tau^*}e^{-\eta\tau^*} = \frac{\gamma_f}{\eta-\alpha}\pi^*\left(\frac{\pi_0}{\pi^*}\right)^{\tilde\beta} = \frac{\gamma_f}{\eta-\alpha}\pi_0\left(\frac{\pi_0}{\pi^*}\right)^{\tilde\beta-1}, \\
\mathfrak{T}^r &= \frac{\gamma_r}{\eta-\alpha}e^{-(\eta-\alpha)\nu}\mathbf{E}\pi_{\tau^*}e^{-\eta\tau^*} = \frac{\gamma_r}{\eta-\alpha}e^{-(\eta-\alpha)\nu}\pi^*\left(\frac{\pi_0}{\pi^*}\right)^{\tilde\beta} \\
&= \frac{\gamma_r}{\eta-\alpha}\pi_0 e^{-(\eta-\alpha)\nu}\left(\frac{\pi_0}{\pi^*}\right)^{\tilde\beta-1}.
\end{aligned}
$$

For proving 3) – 5), let us differentiate both sides of formula (B3) in $\theta$:

$$\mathbf{E}\tau^* e^{-\theta\tau^*} = \left(\frac{\pi_0}{\pi^*}\right)^{\beta(\theta)}\beta'(\theta)\log\left(\frac{\pi^*}{\pi_0}\right).$$

Differentiating the equation $\frac{1}{2}\sigma^2\beta(\theta)(\beta(\theta)-1)+\alpha\beta(\theta)-\theta=0$ in $\theta$, we get $\beta'(\theta)=\left(\sigma^2\beta(\theta)+\alpha-\frac{1}{2}\sigma^2\right)^{-1}$. Therefore:

$$E\tau^* e^{-\theta\tau^*} = \left(\frac{\pi_0}{\pi^*}\right)^{\beta(\theta)}\left(\sigma^2\beta(\theta)+\alpha-\frac{1}{2}\sigma^2\right)^{-1}\log\left(\frac{\pi^*}{\pi_0}\right). \qquad (B6)$$

Note also, that:

$$\mathbf{P}\{\tau^*<\infty\} = \lim_{\theta\to 0}\mathbf{E}e^{-\theta\tau^*}. \qquad (B7)$$

Let us study the behaviour of $\beta(\theta)$ – the positive root of the equation:

$$\beta\left(\alpha-\frac{1}{2}\sigma^2+\frac{1}{2}\sigma^2\beta\right)=\theta \qquad (B8),$$

when $\theta\to 0$.

If $\alpha>\frac{1}{2}\sigma^2$ then, from (B8), it follows that $\beta(\theta)\to 0$ when $\theta\to 0$, while (B7) implies that:

$$\mathbf{P}\{\tau^*<\infty\}=1.$$

Taking a limit to (B6) when $\theta\to 0$, we get $\mathbf{E}\tau^*=(\alpha-\frac{1}{2}\sigma^2)^{-1}\log(\pi^*/\pi_0)$.

If $\alpha=\frac{1}{2}\sigma^2$ then, from (B8), we have $\beta(\theta)=\sqrt{\theta/\alpha}$ and, by virtue of (B6):

$$\mathbf{E}\tau^* \ge \mathbf{E}\tau^* e^{-\theta\tau^*} = \frac{1}{2\sqrt{\theta\alpha}}\left(\frac{\pi_0}{\pi^*}\right)^{\sqrt{\theta/\alpha}}\log\left(\frac{\pi^*}{\pi_0}\right)\to\infty \quad (\theta\to 0).$$

If $\alpha < \frac{1}{2}\sigma^2$, then, due to (B8), $\beta(\theta) \to 1 - 2\alpha/\sigma^2$ and (B7) implies:

$$\mathbf{P}\{\tau^* < \infty\} = \left(\frac{\pi_0}{\pi^*}\right)^{1-2\alpha/\sigma^2} < 1,$$

hence, $\mathbf{E}\tau^* = \infty$.

Theorem 2 is proved completely.

## Notes

1. We will refer to the creation of a  enterprise, not the reconstruction of an  one, because we mean that a  taxpayer will appear.

2. By the initial period of time we mean the moment when the project becomes available for investment.

3. In this study, we define the profit of the firm as its .

4. Partial exemptions do not change the picture cardinally, they just make the formulas more complicated. Consequently, we consider only tax holidays with full exemptions.

5. As a weaker condition, one can consider the case "with a lag", when a project begins to return positive profits after some period of time following the investment.

6. If a set of such $t$ is empty, then put $\tau^* = \infty$.

7. Dependence on the other parameters has a less definite character and can vary depending on the composition of the input parameters.

8. These data were kindly submitted by Prof. S.A.Smolyak on the basis of experience with real investment projects in Russia.

9. In several regions, including Pskov and Tver, Altay territory, and Chuvashia, there is a possibility of extending the holidays up to 10 years.

10. This case was considered in the diploma thesis of S.V. Arkina (Moscow State University, 1997).

11. This notation means that the process $\pi_t$ starts from the deterministic state $\pi_0$.

12. We also accept in general infinite values (with a positive probability).

13. For our problem, this method consists of finding such a function $s(\pi)$ and a level $z^*$ inside the continuation region that they form a solution to the so-called Stefan's problem with a free boundary, namely they satisfy the following Bellman equation:

$$\frac{1}{2}\sigma^2\pi^2 s''(\pi) + \alpha\pi s'(\pi) - \rho s(\pi) = 0$$

and the boundary conditions:

$$s(0) = 0, \quad s(z^*) = G(z^*), \quad s'(z^*) = G'(z^*).$$

As a rule in 'economic' literature (see, e.g. McDonald and Siegel (1986), Dixit and Pindyck (1994), McKean (1965), Merton (1973)), it is supposed without any argument that $s(\pi) = F(\pi)$ and, hence, the optimal stopping point is specified by the level $z^*$. It is not difficult to give an example when there exists a unique solution to Stefan's problem, but it is not a solution to the optimal stopping problem. The general optimal stopping theory proposes some conditions under which Stefan's problem is equivalent to an optimal stopping problem (see Shiryaev (1969)). But, unfortunately, these conditions are very hard to check, and the "smooth pasting" method is considered for the concrete problems to be a heuristic approach to finding a solution which needs additional proof of its optimality (see Shiryaev (1969), Shiryaev et al. (1994)).

## REFERENCES

Black, F., and Scholes, M. (1973): The Pricing of Options and Corporate Liabilities. 3 (May-June), 637–657.

Bond, E. (1981): Tax Holidays and Industry Behaviour. 63 (1), 88–95.

Clark, E. (1997): Valuing Political Risk. 16 (3), 477–490.

Daly, M., Gorman, I., Lenjosek, G., MacNevin, A., and Phiriyapreunt, W. (1993): The Impact of Regional Investment Incentives on Employment and Productivity: Some Canadian Evidence. 23 (4), 559–575.

Dixit, A.K., and Pindyck, R.S. (1994): Princeton University Press, Princeton.

Egorova, E.N., Ternopolskaya, G.B., and Tiutiuriukov, N.N. (1997): Comparison of the Influence of the Tax System on the Investment Activity of

Enterprises in Russia and Foreign Countries. CEMI Russian Academy of Sciences, Moscow (in Russian).

Feltenstein, A., and Shah, A. (1995): General Equilibrium Effects of Investment Incentives in Mexico. 46 (2), 253–269.

Gikhman, I.I., and Skorokhod, A.V. (1977): Nauka, Moscow (in Russian).

Hines, J.R., Jr. (1994): Credit and Deferral as International Investment Incentives. 55 (1), 323–347.

Kueschnigg, C. (1989): Tax Incentives for Investment: A Dynamic General Equilibrium Perspective. 16 (1), 31–51.

MacKie-Mason, J.K. (1990): Some Non-linear Tax Effects on Asset Values and Investment Decisions under Uncertainty. 42 (3), 301–327.

McDonald, R., and Siegel, D. (1986): The Value of Waiting to Invest. 101 (4), 707–727.

McKean, H.P., Jr. (1965): Appendix: A Free Boundary Problem for the Heat Equation Arising from a Problem in Mathematical Economics. 6 (Spring), 32–39.

Merton, R. (1973): Theory of Rational Option Pricing. 4 (Spring), 141–183.

1994, Terinvest, Moscow (in Russian).

Mintz, J.M., and Tsiopoulos, T. (1994): The Effectiveness of Corporate Tax Incentives for Foreign Investment in the Presence of Tax Crediting. 55 (1), 233–255.

Petrakov, N., Arkin, V., and Manevich, V. (1997): Insurance of Investments from Political Risks: Practical Perspectives. April, 16–17 (in Russian).

Samuelson, P.A. (1965): Rational Theory of Warrant Pricing. 6 (Spring), 13–31.

Shiryaev, A.N. (1969): . Nauka, Moscow (in Russian).

Shiryaev, A.N., Kabanov, Yu.M., Kramkov, D.O., and Melnikov, A.V. (1994): Towards the Theory of Pricing of Options of both European and American Types. I. Discrete case, II. Continuous case. 39 (1), 23–129.

(1997). №№1–12. State Committee on Statistics of the Russian Federation, Moscow (in Russian).